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MATH 374 : TOPOLOGY

Problem Set 10 – due Friday, November 22nd

INSTRUCTIONS:

This assignment is due at 4pm on Friday, to be submitted to the mailbox outside my office. (This week there will be no late penalty for submitting Friday.) However, assignments will not be accepted after 4pm on Friday.

10.1 In class, Daniel proposed that $(\mathbb{R} \times \mathbb{R}_{\geq 0})/\sim$ (identifying the horizontal axis to a single point) should be homeomorphic to $\mathbb{R}^2$, and thus, that $\mathbb{R} \times \mathbb{R}_{\geq 0}$ cannot itself be homeomorphic to $\mathbb{R}^2$. Can you find an example of homeomorphic spaces X and Y and a nontrivial equivalence relation $\sim$ on X such that $X/\sim$ is also homeomorphic to Y?

10.2 Prove that normality is preserved by homeomorphism. Is it also preserved under continuous surjections?

10.3 Prove that any metric space is normal.

10.4 Prove that $\mathbb{R}_{\text{sorgenfrey}}$ is normal. [*Hint: Given two closed sets, find an open cover of each.*]

10.5 The goal of this exercise is to establish that $(\mathbb{Z}, \mathcal{T}_{\text{Furstenberg}})$ is *metrizable*, i.e. that there exists a metric on $\mathbb{Z}$ that induces the Furstenberg topology. (In fact, there exist many such metrics!) Set

$$d(a,b) = 1 - \sum_{n|a-b} \frac{1}{2^n}$$

where the sum runs over all positive integers n dividing $a - b$. For example, $d(2,6) = \frac{3}{16}$.

(a) Prove that d is a metric on $\mathbb{Z}$.
(b) Prove that any nonempty open ball with respect to d contains a bi-infinite arithmetic progression.
(c) Prove that any bi-infinite arithmetic progression contains a nonempty open ball with respect to d.
(d) Prove that the metric d induces the Furstenberg topology on $\mathbb{Z}$.

10.6 Recall that a space is *Hausdorff* iff any two distinct points live in disjoint opens; a space is *normal* iff any two disjoint closed sets can be enlarged to two disjoint open sets. There's a natural property in between Hausdorffness and normality:

Definition. A space is *regular* iff for any point p and any closed set C not containing p, there exist disjoint open sets $\mathcal{O}$ and $\mathcal{O}'$ such that $p \in \mathcal{O}$ and $C \subseteq \mathcal{O}'$.

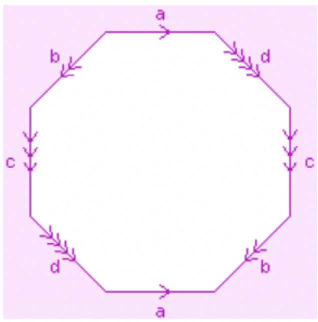
The purpose of this exercise is to explore the relationships between our various separation axioms.

(a) Show that if a space is both regular and T_0 then it must be Hausdorff.
(b) Give an example of a regular space that's not Hausdorff.
(c) Prove that if X is compact and Hausdorff then it must be regular. [*Hint: Given a point p and a closed set C, try to separate p from every point of C.*]

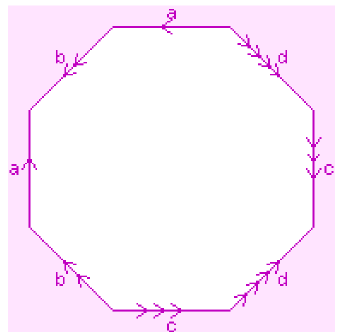
(d) Prove that if X is compact and Hausdorff then it must be normal.

(e) If a space is both normal and T_1, must it be compact and Hausdorff?

10.7 Give the simplest description you can of the surfaces represented by each of the following gluing diagrams. (You don't have to rigorously prove anything.)



Gluing diagram (a)



Gluing diagram (b)