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MATH 374 : TOPOLOGY

Solution Set 4

4.1 The topology of $\mathbb{R}^2$ induced by the Euclidean metric is called the *usual topology* on $\mathbb{R}^2$. Describe the topology of $\mathbb{R}^2$ induced by the taxicab metric. What about the chessboard metric? What about the British Rail metric?

Taxicab. The topology of $\mathbb{R}^2$ induced by the taxicab metric is the usual topology on $\mathbb{R}^2$. In problem **2.5** you proved that the taxicab topology is a refinement of the usual topology, so it suffices to prove that the usual topology is a refinement of the taxicab topology. To prove this, it's convenient to introduce a bit of notation: let $\mathcal{B}^e_r(\alpha)$ denote the Euclidean ball of radius r around α, and $\mathcal{B}^t_r(\alpha)$ the taxicab ball of radius r around α. I claim:

Lemma 1. *For any $r > 0$, $\mathcal{B}^e_{r/10}(\alpha) \subseteq \mathcal{B}^t_r(\alpha)$.*

Once we prove this, it's not hard to show that the taxicab and usual metrics produce the same topology. Indeed, if A is open with respect to the taxicab metric, then every point $\alpha \in A$ is interior in A with respect to the taxicab metric, so there's an open taxicab ball around α that's contained entirely in A. By the lemma, there's an open euclidean ball strictly inside the taxicab ball, which means that $\alpha \in \text{int}(A)$ with respect to the euclidean metric as well.

Proof of Lemma. Suppose $x \in \mathcal{B}^e_{r/10}(\alpha)$. The Cauchy-Schwarz inequality (see document posted on course website) implies

$$\big(|x_1 - \alpha_1| + |x_2 - \alpha_2|\big)^2 \leq 2\big(|x_1 - \alpha_1|^2 + |x_2 - \alpha_2|^2\big) \leq \frac{r^2}{50},$$

whence $|x_1 - \alpha_1| + |x_2 - \alpha_2| \leq r$. This is equivalent to $x \in \mathcal{B}^t_r(\alpha)$. ☐

Discussion. Visually, open balls with respect to the taxicab metric are diamonds; by appropriately rescaling these, we can make them fit inside any given euclidean ball, and also make them large enough to contain any particular euclidean ball. The proof above simply formalizes this. Note that the choice of $\frac{r}{10}$ is lazy—it could be sharpened. That said, sharpening this requires more thought and doesn't change the conclusion, so why bother!

Chessboard. The topology of $\mathbb{R}^2$ induced by the chessboard metric is the usual topology on $\mathbb{R}^2$. The argument is very similar to the above, except that the chessboard ball of radius r (whose shape is a square) strictly contains the euclidean ball of radius r, which strictly contains the chessboard ball of radius $\frac{r}{10}$.

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British Rail metric. The topology of $\mathbb{R}^2$ induced by the British Rail metric is an odd hybrid of the discrete and euclidean topologies: it is generated by the basis consisting of all singletons other than $\{\mathbf{0}\}$, and all euclidean open balls centered at $\mathbf{0}$. More explicitly, this topology can be expressed in the form

$$\mathcal{T} = \{X \subseteq \mathbb{R}^2 : \mathbf{0} \notin X\} \cup \{X \subseteq \mathbb{R}^2 : \exists \delta > 0, \mathcal{B}_\delta(\mathbf{0}) \in X\}$$

where $\mathcal{B}_\delta(p)$ denotes the euclidean open ball of radius δ around p and $\mathbf{0}$ denotes the origin.

To see this, observe that for any $p \in \mathbb{R}^2$, every other point is at least a distance $|p|$ away (since one must first travel to $\mathbf{0}$). It follows that if $p \neq 0$, the open British Rail ball around p of radius $\frac{|p|}{2}$ consists of only the point p. On the other hand, open balls around $\mathbf{0}$ in the British Rail metric agree with euclidean open balls around $\mathbf{0}$.

4.2 Note that the basis we gave for $\mathbb{R}_{\text{usual}}$ (the collection of all bounded open intervals) has uncountably many sets in it. Find a *countable* basis of $\mathbb{R}_{\text{usual}}$.

I claim that

$$\mathcal{B} := \{(a, b) : a, b \in \mathbb{Q}\}$$

is a countable basis of $\mathbb{R}_{\text{usual}}$. It's clearly countable, so it suffices to show that any element of the basis we gave in class is a union of elements of $\mathcal{B}$. To see this, suppose (α, β) is an arbitrary bounded open interval. Pick a sequence of rationals (a_n) such that $a_n \geq \alpha$ for all n and $a_n \to \alpha$; pick another sequence of rationals (b_n) such that $b_n \leq \beta$ for all n and $b_n \to \beta$. Then $(\alpha, \beta) = \bigcup_{n \geq 1} (a_n, b_n)$.

4.3 Suppose $\mathcal{T}$ is a topology on $\mathbb{R}^2$ that contains the set of points $\{(x, x) : x \in \mathbb{R}\}$, and also contains the line segments $(x, x + 2) \times \{y\}$ for each $x, y \in \mathbb{R}$. (*Here* (x, x) *denotes a point in the plane, while* $(x, x + 2)$ *denotes an open interval.*)

(a) Is the interval $(\frac{3}{4}, 1) \times \{0\} \in \mathcal{T}$?

Yes: both $(-1, 1) \times \{0\}$ and $(3/4, 11/4) \times \{0\}$ are open, so their intersection must be as well.

(b) Is the interval $(1, 4) \times \{0\} \in \mathcal{T}$?

Yes: both $(1, 3) \times \{0\}$ and $(2, 4) \times \{0\}$ are open, so their union must be as well.

(c) Does $\mathcal{T}$ contain an element consisting of **countably** infinitely many points?

Yes. Observe that for any $n \in \mathbb{Z}$, the singleton $\{(n, n)\} \in \mathcal{T}$, since it's the intersection of the open sets $\{(x, x) : x \in \mathbb{R}\}$ and $(n - 1, n + 1) \times \{n\}$. It follows that $\bigcup_{n \in \mathbb{Z}} (n, n) \in \mathcal{T}$.

4.4 Let $\overline{\,\cdot\,}$ be a closure operator on X. Prove that $A \subseteq B \subseteq X$ implies $\overline{A} \subseteq \overline{B}$. [*You may not use properties of closed sets for this problem, since we used this as a lemma in class to prove properties of closed sets!*]

Observe that $\overline{A} \cup \overline{B \setminus A} = \overline{A \cup (B \setminus A)} = \overline{B}$, whence $\overline{A} \subseteq \overline{B}$.

4.5 In class, Daniel made the very reasonable proposal that the closure of a singleton set (i.e. a set consisting of a single element) should be itself. Sadly, topology cares little for our intuition.

Construct an example of a topological space $(X, \mathcal{T})$—i.e. a space X with a topology $\mathcal{T}$ satisfying our three conditions from class—in which $2 \leq |X| < \infty$, and $\overline{\{x\}} \neq \{x\}$ for some $x \in X$. Can you construct such an example in which X is infinite?

Take any set X containing at least two elements, and consider $(X, \mathcal{T}_{\text{indiscrete}})$. Since the only nonempty closed set is X, the closure of any singleton must be all of X.

Discussion. In the finite case, it's possible to construct topologies by hand in which singletons aren't closed. For example, let $X = \{a, b, c, d\}$ and consider $\mathcal{T} = \{\varnothing, X, \{c\}, \{d\}, \{c, d\}\}$; this is easily verified to be a topology on X. The smallest closed set containing b is $\{a, b\}$, so $\overline{\{b\}} = \{a, b\}$.

4.6 Prove that the intersection of any collection of topologies on X is a topology on X.

Let $\mathcal{T} := \bigcap_{\alpha} \mathcal{T}_\alpha$ be the intersection of a bunch of topologies on X. Clearly $\varnothing, X \in \mathcal{T}$, since they are both elements of each $\mathcal{T}_\alpha$, so it suffices to show that $\mathcal{T}$ is closed under finite intersections and arbitrary unions. But both of these hold in every $\mathcal{T}_\alpha$, so they are automatically inherited in $\mathcal{T}$.

4.7 Recall from class that given any closure operator $\overline{\,\cdot\,}$ on X, we can define what it means for a set to be closed: we say $A \subseteq X$ is *closed* iff $A = \overline{A}$. Let $\mathcal{C}$ denote the collection of all subsets of X that are closed (with respect to a given closure operator). In class we proved that $\mathcal{C}$ must satisfy three properties:

(C-i) $\emptyset, X \in \mathcal{C}$,

(C-ii) $\mathcal{C}$ is closed under finite unions, and

(C-iii) $\mathcal{C}$ is closed under arbitrary intersections.

[NB: the word *closed* in (ii) and (iii) is unrelated to the term *closed set*!]

(a) In class we asserted that the process can be run in reverse, as well: given a set $\mathcal{C} \subsetneq \mathcal{P}(X)$, we can define a closure operator on X by setting the closure of S to be the smallest element of $\mathcal{C}$ containing S. Prove that if $\mathcal{C}$ fails to satisfy any one of (C-i), (C-ii), or (C-iii), then the induced 'closure' can fail to be a closure.

Let $X := \{1, 2, 3, 4\}$.

Consider $\mathcal{C} = \{\emptyset\}$. This violates only $X \in \mathcal{C}$. However, it has $\overline{\{1\}} = \emptyset$, violating a closure property since $\{1\} \not\subseteq \overline{\{1\}}$. Consider $\mathcal{C} = \{X\}$. This violates only $\emptyset \in \mathcal{C}$. However, it has $\overline{\emptyset} = X \neq \emptyset$, violating a closure property.

Next, consider $\mathcal{C} = \{\emptyset, \{1\}, \{2\}, X\}$. It violates only (C-ii), but fails one of the closure properties:

$$\overline{\{1\}} \cup \overline{\{2\}} = \{1\} \cup \{2\} = \{1, 2\} \neq X = \overline{\{1\} \cup \{2\}}.$$

Finally, consider $\mathcal{C} = \{\emptyset, \{1, 2\}, \{1, 3\}, X\}$. This violates only (C-iii), but fails a closure property:

$$\overline{\{1, 2\}} \cup \overline{\{1, 3\}} = \{1, 2\} \cup \{1, 3\} = \{1, 2, 3\} \neq X = \overline{\{1, 2\} \cup \{1, 3\}}.$$

(b) Verify that if $\mathcal{C}$ does satisfy all of (C-i), (C-ii), and (C-iii), then the induced operator described above *is* a closure.

We verify each property of closure:

- Since $\emptyset \in \mathcal{C}$, $\overline{\emptyset}$ is defined as an intersection involving an empty set, hence is empty.
- For an arbitrary set A, its closure, defined as an intersection of sets in $\mathcal{C}$, is itself in $\mathcal{C}$. Taking the closure of $\overline{A}$ again gives an intersection of sets involving $\overline{A}$ itself, which returns $\overline{A}$.
- For an arbitrary set A, its closure is an intersection of at least one set, since $X \in \mathcal{C}$. Since each set in the intersection is a superset of A, so is their intersection $\overline{A}$.
- To show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$, we want

$$\bigcap_{\substack{X \supset (A \cup B) \\ X \in C}} X = \bigcap_{\substack{Y \supset A \\ Y \in C}} Y \cup \bigcap_{\substack{Z \supset B \\ Z \in C}} Z.$$

Since every X also appears as one of the Y's, the intersection of Y's is an intersection involving more sets than the intersection of X's, which means $\bigcap_{\substack{Y \supset A \\ Y \in C}} Y \subseteq \bigcap_{\substack{X \supset (A \cup B) \\ X \in C}} X$.

Applying the same logic to the Z's gives $\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$.

For the other direction, we will show that if a point x is not in $\overline{A} \cup \overline{B}$, then x is not in $\overline{A \cup B}$. Suppose x is not in $\bigcap_{\substack{Y \supset A \\ Y \in C}} Y \cup \bigcap_{\substack{Z \supset B \\ Z \in C}} Z$. This means x is in neither intersection, which means there exists $Y' \supset A, Z' \supset B, Y', Z' \in C$ both excluding x. Since C is closed under finite unions, the set $Y' \cup Z'$ is some X' in the intersection defining $\overline{A \cup B}$, which means $x \notin \overline{A \cup B}$ as desired.

(c) Prove that if you start with a closure, generate $\mathcal{C}$ as described above, and then use $\mathcal{C}$ to induce a closure, you end up with the same closure operator you started with.

Beginning with a closure $\overline{\cdot}$, we generate the set

$$\mathcal{C} := \{\overline{S} : S \in \mathcal{P}(X)\},$$

and then make the function

$$f(A) = \bigcap_{A \subseteq \overline{S}, \overline{S} \in \mathcal{C}} \overline{S}.$$

To show $f(A) = \overline{A}$, note that $A \subseteq \overline{A}$, so $\overline{A}$ is an element of $\mathcal{C}$ containing A. We must now prove that $\overline{A}$ is the smallest element of $\mathcal{C}$ containing A. Assume to the contrary that there is some $S \in \mathcal{C}$ such that $A \subseteq S$ and $\overline{A} \not\subseteq S$. Let $R := S \backslash A$. We now have

$$\overline{R \cup A} = \overline{R} \cup \overline{A}.$$

As $R \cup A = S \in \mathcal{C}$, the first closure is just itself. Thus

$$S = \overline{R} \cup \overline{A}.$$

However, $\overline{A} \not\subseteq S$, a contradiction.

(d) Prove that if you start with a set $\mathcal{C}$ satisfying properties (C-i), (C-ii), and (C-iii), induce a closure, and then use that closure to induce a set of closed sets, this set is $\mathcal{C}$.

Given $\mathcal{C}$ satisfying (i), (ii), (iii), define the closure $\overline{A} = \bigcap_{\substack{S \supset A \\ S \in \mathcal{C}}} S$. We want to show $A = \overline{A}$ iff $A \in \mathcal{C}$. If $A \in \mathcal{C}$, then A is in the intersection defining $\overline{A}$; since every set in the intersection contains A, the intersection equals A. For the other direction, suppose $A = \overline{A} = \bigcap_{\substack{S \supset A \\ S \in \mathcal{C}}} S$. Since A is expressible as an intersection of sets in $\mathcal{C}$, A is in $\mathcal{C}$ by (iii).

(e) Deduce that there's a bijection between the set of all possible closures on X and the set of all possible topologies on X. (Your proof should be quite short.)

Since the set complement is its own inverse operator, the topologies on X are in bijection with the collections of closed sets, i.e., the possibilities of $\mathcal{C}$. Let f be the map from the set of possible closures to the set of possible collections of closed sets, defined by inducing closed sets from closure. Let g be the map from the set of possible collections of closed sets to the set of possible closures defined by inducing closure from closed sets. By c), $g \circ f$ is the identity; in particular, f is injective. By d), $f \circ g$ is the identity; in particular, f is surjective. Thus f is a bijection between the collection of all possible collections of closed sets and the set of all possible closures.

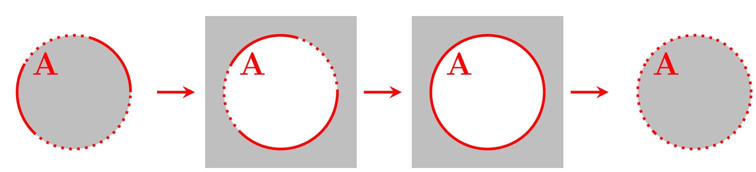
4.8 Given a set X and a closure operator $\overline{\,\cdot\,}$ on X. One of the defining properties of closure is that it's *idempotent*: applying it repeatedly produces the same result as applying it once. In particular, starting with a set $A \subseteq X$ one can generate at most 2 distinct sets using the closure operator: A and $\overline{A}$. The purpose of this exercise is to explore the relationship between the closure and complement operators.

(a) Do the closure and complement operators commute? In other words, given $A \subseteq X$, does $\overline{A^c} = \overline{A}^{\,c}$?

No. For example, in $\mathbb{R}_{\text{usual}}$, $\overline{\{1\}^c} = \mathbb{R} \neq \mathbb{R} \setminus \{1\} = \overline{\{1\}}^{\,c}$.

(b) Define a new operator $i : \mathcal{P}(X) \to \mathcal{P}(X)$ by $i(A) := \left(\overline{A^c}\right)^c$. Can you give an intuitive description of this set? (Think in $\mathbb{R}^2$!)

$i(A)$ is the interior of A. Here's a picture:



(c) Given $A \subseteq X$, prove that there are only finitely many different sets that can be generated from A by applying complements and closures. Can you get an explicit upper bound on how many? [*Hint: Your upper bound shouldn't depend on A or X!*]

Discussion. It's tempting to assume that

$$\overline{i(A)} = \overline{A}\,.$$

This seems obvious, but it is false even in familiar topological spaces. For example, consider any singleton set $A \subset \mathbb{R}_{\text{usual}}$; the left hand side is $\varnothing$, while the right hand side is a singleton.

We can create at most 14 sets using complements and closures, and the number 14 cannot be decreased! This was originally published in 1922 by Kuratowski.

Let cS denote the complement of S, and kS denote the closure of S. Since $kkS = kS$ and $ccS = S$, every set that can be formed out of A using just closures and complements will be of the form of strictly alternating applications of c and k. We will show that $kckckckA = kckA$, which immediately implies that there are at most 14 sets we can generate:

$$A, kA, cA, ckA, kcA, ckcA, kckA, ckckA,$$
$$kckcA, ckckcA, kckckA, ckckckA, kckckcA, ckckckcA.$$

Remarkably, it is possible to choose A so that all 14 of the above are distinct sets; check out this website.

First we show $kckckckA \supseteq kckA$. From part (b), $ckcA = \text{int}(A)$, whence

$$ckc(kA) = \text{int}(kA) \subseteq kA.$$

Note that in Problem **4.4** we showed that if $A \subseteq B$ then $kA \subseteq kB$. So,

$$\begin{aligned} & k(ckc(kA)) \subseteq kkA = kA \\ \Rightarrow\ & c(k(ckc(kA))) \supseteq ckA \\ \Rightarrow\ & k(c(k(ckc(kA)))) \supseteq kckA \qquad \text{by } \mathbf{4.4} \end{aligned}$$

Next we show $kckckckA \subseteq kckA$:

$$\begin{aligned} & ckc(kckA) \subseteq kckA \qquad \text{by part (b)} \\ \Rightarrow\ & k(ckc(kckA)) \subseteq kkckA = kckA \qquad \text{by } \mathbf{4.4} \end{aligned}$$

Whew!