Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 374 : TOPOLOGY

Solution Set 5

5.1 In Chapter 2 of Ivan's notes (*Bases of topologies*), a basis on X is defined to be any set $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfying two conditions:

- $\mathcal{B}$ covers X
- For any $S, T \in \mathcal{B}$ and any $\alpha \in S \cap T$, $\exists A \in \mathcal{B}$ such that $\alpha \in A \subseteq S \cap T$.

The first condition is identical to the definition we gave in class, but the second looks different. Prove that this is equivalent to the definition we gave in class.

First, we show that our class' definition implies Ivan's. Fix any $S, T \in \mathcal{B}$ and any $\alpha \in S \cap T$. By our definition, $S \cap T$ is a union of basis elements, whence α must live in at least one of these; call it A. Then $\alpha \in A \subseteq S \cap T$, as claimed.

Next we show that Ivan's definition implies ours. Pick any two basis elements $S, T \in \mathcal{B}$; we wish to show that $S \cap T$ is a union of basis elements. From Ivan's definition we know that for any $\alpha \in S \cap T$ there exists $A \in \mathcal{B}$ such that $\alpha \in A \subseteq S \cap T$. It follows that $\boxed{S \cap T \subseteq \bigcup_{A \in \mathcal{A}} A}$, where $\mathcal{A}$ is a collection of elements of $\mathcal{B}$ that are also subsets of $S \cap T$. On the other hand, pick any $\beta \in \bigcup_{A \in \mathcal{A}} A$. Then β is an element of some $A \in \mathcal{A}$, which implies that $\beta \in S \cap T$. It follows that $\boxed{\bigcup_{A \in \mathcal{A}} A \subseteq S \cap T}$. Combining the two boxed expressions yields the claim.

5.2 Prove that the topology defined by Definition 2.6 in Chapter 2 of Ivan's notes is the same as the topology generated by $\mathcal{B}$, as defined in our class.

First, recall our class' definition:

$$\mathcal{T} = \{\mathcal{O} \subseteq X : \mathcal{O} = \bigcup_{\alpha} B_\alpha \text{ for some } B_\alpha \in \mathcal{B}\}.$$

And here's Ivan's definition:

$$\mathcal{T}' = \{\mathcal{O} \subseteq X : \forall x \in \mathcal{O}, \exists B \in \mathcal{B} \text{ with } x \in B \subseteq \mathcal{O}\}.$$

We will prove equality of $\mathcal{T}'$ and $\mathcal{T}$ by showing containment in both directions.

($\mathcal{T}' \subseteq \mathcal{T}$) Pick some $\mathcal{O} \in \mathcal{T}'$. Then we know that for all $x \in \mathcal{O}$ there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq \mathcal{O}$. It's straightforward to check verify that $\mathcal{O} = \bigcup_{x \in \mathcal{O}} B_x$. Since each $B_x \in \mathcal{B}$, we deduce $\mathcal{O} \in \mathcal{T}$.

($\mathcal{T}' \supseteq \mathcal{T}$) Pick some $\mathcal{O} \in \mathcal{T}$. We want to show that $\mathcal{O} \in \mathcal{T}'$. Since $\mathcal{O} \in \mathcal{T}, \mathcal{O} = \bigcup B_\alpha$. So pick any $x \in \mathcal{O}$, and we get that $x \in B_\alpha$ for some α. This implies $\mathcal{O} \in \mathcal{T}'$, and we are done.

5.3 Prove that collection of all open balls in $\mathbb{R}^2$—i.e. all sets of the form $\mathcal{B}_\delta(x)$, with respect to the Euclidean metric—is a basis on $\mathbb{R}^2$, and that it generates the usual topology $\mathbb{R}^2$.

Let $\mathcal{B}$ denote the collection of all open balls in $\mathbb{R}^2$.

Claim. *$\mathcal{B}$ is a basis on $\mathbb{R}^2$.*

Proof. $\mathcal{B}$ clearly covers $\mathbb{R}^2$, since every point in $\mathbb{R}^2$ is the center of an open ball. Thus it suffices to verify the second condition of being a basis. We use Ivan's definition: we must show that every point in the intersection of two balls must live inside some ball that's entirely contained in the intersection.

Given $S, T \in \mathcal{B}$, pick $x \in S \cap T$. Since $x \in S$ and S is an open ball, there exists an open ball $\mathcal{B}_{\delta_1}(x)$ with $\delta_1 > 0$ such that $x \in \mathcal{B}_{\delta_1}(x) \subseteq S$. Similarly for T, there exists an open ball $\mathcal{B}_{\delta_2}(x)$ with $\delta_2 > 0$ such that $x \in \mathcal{B}_{\delta_2}(x) \subseteq T$. Then $x \in \mathcal{B}_{\min(\delta_1,\delta_2)}(x) \subseteq S \cap T$. We've verified Ivan's second condition, and have therefore shown that $\mathcal{B}$ is a basis on $\mathbb{R}^2$. □

Claim. *$\mathcal{B}$ generates the usual topology.*

Proof. Note that any union of open balls is an element of the usual topology: every point in this union is an interior point, since it lives in one of the open balls composing this union. It thus suffices to prove that every element of the usual topology is generated by $\mathcal{B}$.

Consider a set $\mathcal{O}$ in the usual topology in $\mathbb{R}^2$. Every $x \in \mathcal{O}$ is an interior point, meaning there exists an open ball B such that $x \in B \subseteq \mathcal{O}$. By Ivan's definition, $\mathcal{O}$ is an element of the topology generated by $\mathcal{B}$. □

5.4 All the parts of this question concern the Sorgenfrey line.

(a) Prove that the interval $(0, 1)$ is open.

Observe that $(0,1) = \bigcup_{n=1}^{\infty} [1/n, 1)$ is a union of basis elements, hence is open.

(b) Is $(0, 1]$ open?

No. In fact, I claim that any open set containing 1 must also contain some $\gamma > 1$.

Suppose $\mathcal{O}$ is some open set containing 1. Since we can write $\mathcal{O}$ as a union of basis elements, 1 must live in one of these basis elements $[\alpha, \beta)$, i.e. $\alpha \leq 1 < \beta$. But then $\frac{1+\beta}{2}$, which is strictly larger than 1, lives in $[\alpha, \beta) \subseteq \mathcal{O}$ as well.

(c) Prove that singletons are their own closures.

PROOF 1. Pick any $x \in \mathbb{R}$; I claim that $\{x\}$ is closed. This is equivalent to showing that its complement is open. We accomplish this by exhibiting it as a union of basis elements:

$$\mathbb{R} \setminus \{x\} = (-\infty, x) \cup (x, \infty) = \left(\bigcup_{n=1}^{\infty} [x-n, x) \right) \cup \left(\bigcup_{n=1}^{\infty} [x + \frac{1}{n}, x+n) \right).$$

This proves that $\{x\}$ is closed, hence is its own closure. □

PROOF 2. The argument given in part (a) applies to any open interval, and therefore implies that the lower limit topology is a refinement of the usual topology. It follows that all sets that are closed in the usual topology must also be closed in the lower limit topology; in particular, singletons must be closed. □

(d) Prove that there does not exist a countable basis that generates the lower limit topology. [*Hint: In view of problem 4.2, make sure your proof doesn't apply to* $\mathbb{R}_{usual}$.]

Suppose $\mathcal{B}$ is a basis of the lower limit topology.

Claim. *For any* $x \in \mathbb{R}$*, there exists* $y > x$ *such that* $[x, y) \in \mathcal{B}$.

Of course, once we prove this we're done, since there are uncountably many $x \in \mathbb{R}$.

Proof of claim. Given any $x \in \mathbb{R}$, we know $[x, x+1)$ is open. Since this is expressible as a union of elements of $\mathcal{B}$, there must exist an element of $\mathcal{B}$ that contains x. However, that element cannot contain anything strictly smaller than x. □

5.5 Let $\mathcal{B}$ be the set of bi-infinite arithmetic progressions consisting of integers, and let $\mathcal{T}$ denote the topology on $\mathbb{Z}$ generated by $\mathcal{B}$. (This is the *Furstenberg topology* on $\mathbb{Z}$ that we used to prove the infinitude of primes.)

(a) Prove that $\mathcal{B}$ is a basis on $\mathbb{Z}$.

We have $\mathcal{B} = \{a\mathbb{Z} + b : a, b \in \mathbb{Z}, a \neq 0\}$. In particular, $\mathbb{Z} \in \mathcal{B}$, so $\mathcal{B}$ covers $\mathbb{Z}$. Next, given any two elements of $\mathcal{B}$, say $B_1 := a_1\mathbb{Z} + b_1$ and $B_2 := a_2\mathbb{Z} + b_2$, pick any $n \in B_1 \cap B_2$. Since $n \in a_1a_2\mathbb{Z} + n \subseteq B_1 \cap B_2$, Ivan's second condition implies $\mathcal{B}$ is a basis.

(b) Let $a_n := 2^n 3^{n-1} 5^{n-2} \cdots p_{n-1}^2 p_n$, where p_k denotes the k^{th} prime number; the sequence a_n begins $2, 12, 360, 75600, \ldots$ Does this sequence converge in $\mathbb{Z}$ under the Furstenberg topology? If not, prove it; if so, find all values it converges to, and prove that your list is exhaustive.

Claim. $a_n \to 0$*. Moreover,* a_n *doesn't converge to anything else.*

Proof. Pick any open set $\mathcal{O}$ containing 0. Since $\mathcal{O}$ is a union of basis elements, there must be a bi-infinite arithmetic progression $d\mathbb{Z} + k$ such that $0 \in d\mathbb{Z} + k \subseteq \mathcal{O}$. It instantly follows that $d \mid k$, whence $d\mathbb{Z} + k = d\mathbb{Z}$. Now observe that for all large n we have $d \mid a_n$. But this means $a_n \in d\mathbb{Z}$ for all large n, whence $a_n \in \mathcal{O}$ for all large n. This proves that $a_n \to 0$.

Next, pick any $\ell \neq 0$. We wish to show that a_n doesn't converge to ℓ. Consider the set $B := -1 + (\ell + 1)\mathbb{Z}$. This is open (it's an element of the basis) and it contains ℓ. Observe that the $a_n \notin B$ for all large n, since $a_n \equiv 0 \pmod{\ell + 1}$ for all large n while all the elements of B are $\equiv -1 \pmod{\ell + 1}$. □

5.6 In class, I asserted without proof that a space is T_1 iff singletons are closed. In fact, more is true! Given a topological space $(X, \mathcal{T})$, prove that

$$(X, \mathcal{T}) \text{ is } T_1 \iff \{x\} \text{ is closed } \forall x \in X \iff \text{all finite sets are closed} \iff \forall A \subseteq X, A = \bigcap_{\substack{\mathcal{O} \text{ s.t.} \\ A \subseteq \mathcal{O} \in \mathcal{T}}} \mathcal{O}$$

Proposition 1. *X is* T_1 *iff every singleton is closed.*

Proof. ($\Rightarrow$) Suppose X is T_1. Pick any $p \in X$. For every $x \in X \setminus \{p\}$, there exists an open set $\mathcal{O}_x$ that contains x but not p. This implies $X \setminus \{p\} = \bigcup_{x \neq p} \mathcal{O}_x$, whence $X \setminus \{p\}$ is a union of open sets. We deduce that $X \setminus \{p\}$ is open, so the singleton $\{p\}$ is closed.

($\Leftarrow$) Given points $\alpha \neq \beta$ in X. By hypothesis both $\{\alpha\}$ and $\{\beta\}$ are closed, whence $\{\alpha\}^c$ and $\{\beta\}^c$ are both open; moreover, each of these opens contains one point but not the other. Since α, β were arbitrary, we deduce X is a T_1 space. □

continued on next page...

Proposition 2. *All singleton sets are closed iff all finite sets are closed.*

Proof. This is immediate, since finite unions of closed sets are closed. □

Proposition 3. *X is T_1 iff every set is equal to the intersection of all opens containing it.*

Proof. ($\Rightarrow$) Given $A \subseteq X$. For any $x \in A^c$ and any $a \in A$ there exists an open set $\mathcal{O}_a$ with $a \in \mathcal{O}_a$ and $x \notin \mathcal{O}_a$. It follows that $U_x := \bigcup_{a \in A} \mathcal{O}_a$ is an open set containing all of A and not containing x. This implies that $A = \bigcap_{x \in A^c} U_x$, so A is the intersection of a bunch of open sets containing it; adding more sets to the intersection won't change this, so A is the intersection of all open sets containing it.

($\Leftarrow$) Suppose X is not T_1. Then there exist distinct points $\alpha, \beta \in X$ such that every open set containing α also contains β. But then the singleton set $\{\alpha\}$ isn't the intersection of all open sets containing it, since this intersection also contains β! □

5.7 All parts of this question concern $\mathbb{R}_{\text{cofinite}}$.

(a) Let $a_n := 1$ for all n. What's $\lim_{n\to\infty} a_n$?

From the definition of convergence, it's clear that $a_n \to 1$. The question therefore becomes: given $\alpha \neq 1$, is it possible that $a_n \to \alpha$? Suppose it does. Given an open set $\mathcal{O} \ni \alpha$, define $U = \mathcal{O} \setminus \{1\}$. Note that U is also open and contains α, but doesn't contain any of the a_n; it follows that a_n does not converge to α. Therefore, a_n converges to 1.

(b) Let $b_n := (-1)^n$ for all n. What's $\lim_{n\to\infty} b_n$?

b_n cannot converge to any $\beta \neq \pm 1$, since the open set $\{\pm 1\}^c$ contains β but not any of the b_n. b_n cannot converge to 1 because $-1 \notin \{-1\}^c$, and the analogous argument applies for -1. Therefore, b_n diverges.

(c) Let $c_n := 1 + 2 + 3 + \cdots + n$. Prove that $c_n \to -\frac{1}{12}$.

Pick any open set $\mathcal{O} \ni -\frac{1}{12}$. Observe that all the c_n are distinct, so $\mathcal{O}$ must contain all but finitely many of them; in particular, it must contain a tail of the sequence. The claim follows.

(d) Can you give a simple and complete description of convergence in $\mathbb{R}_{\text{cofinite}}$?

If (a_n) is eventually constant, then it converges to exactly one value. If no tail of (a_n) is constant, but the set $\{a_n : n \geq 1\}$ is finite, then (a_n) diverges. If the set $\{a_n : n \geq 1\}$ is infinite, then (a_n) converges to every real number.