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MATH 374 : TOPOLOGY

Solution Set 6

6.1 In class we proved that if a topological space is Hausdorff, then every convergent sequence has a unique limit. The goal of this exercise is to show that the converse of this fails to hold.

(a) Prove that in $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$, every convergent sequence has a unique limit.

Lemma 1. *The only convergent sequences in this space are eventually constant.*

Proof. Suppose (a_n) is a convergent sequence, say, $a_n \to L$. Consider the set

$$(\mathbb{R} \setminus \{a_n : n \geq 1\}) \cup \{L\}.$$

This is an open set containing L, so it must contain a tail of the sequence (a_n). This is only possible if $a_n = L$ for all large n. □

It therefore remains to prove that an eventually constant sequence has a unique limit. Suppose $a_n = L$ for all large n, and pick $\alpha \neq L$. Then the open set $\mathbb{R} \setminus \{L\}$ contains α, but doesn't contain the tail of (a_n). It follows that (a_n) doesn't converge to α.

(b) Prove that $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ isn't Hausdorff.

I claim any two nonempty open sets intersect. To see this, say A and B are both nonempty and open; by definition, A^c and B^c are both countable. Then $(A \cap B)^c = A^c \cup B^c$ is countable, hence $A \cap B$ must be uncountable (in particular, nonempty!).

6.2 Given a topology $\mathcal{T}$ on $\mathbb{R}$. Does there exist an open map $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T})$ that's not continuous?

Yes, there are many such! Here's one nice example. Let

$$\mathcal{T} := \{\varnothing, \{0\}, \{1\}, \{0, 1\}, \mathbb{R}\};$$

this is easily verified to be a topology. Consider the function

$$\delta_0(x) := \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\delta_0(\varnothing) = \varnothing$, $\delta_0(0) = 1$, $\delta_0(1) = 0$, and $\delta_0(\mathbb{R}) = \{0, 1\}$, so δ_0 is an open map. However, $\delta_0^{-1}(0) = \mathbb{R} \setminus \{1\}$ is not in $\mathcal{T}$. Thus, δ_0 is open but not continuous.

6.3 Let $\mathcal{T}_{13}$ denote the particular point topology (with respect to 13) on $\mathbb{R}$. What can you say about continuous functions $(\mathbb{R}, \mathcal{T}_{13}) \to \mathbb{R}_{\text{usual}}$? Try to give as simple a description of all such functions as possible (with proofs, of course!).

The only continuous functions from $(\mathbb{R}, \mathcal{T}_{13}) \to \mathbb{R}_{\text{usual}}$ are constant functions. To see this, observe that for any nonconstant function f, $\exists p \in f(\mathbb{R})$ with $13 \notin f^{-1}(p)$. Let B be an open ball around p small enough that it doesn't contain $f(13)$. Then $f^{-1}(B)$ isn't open, hence f isn't continuous.

6.4 Recall from class that given a topological space $(X, \mathcal{T})$, any subset $A \subseteq X$ inherits a natural topology, called the *subspace topology* on A:

$$\mathcal{T}_{\text{subspace}} := \{\mathcal{O} \cap A : \mathcal{O} \in \mathcal{T}\}.$$

Show that the subspace topology A inherits from X is the coarsest topology on A such that i is continuous on A, where $i : A \to X$ is defined $i(x) := x$.

Let $\mathcal{T}'$ be a topology on A such that i is continuous on A. Observe that $i^{-1}(\mathcal{O}) = \mathcal{O} \cap A$ for any $\mathcal{O} \in \mathcal{T}$, whence $\mathcal{O} \cap A \in \mathcal{T}'$. It follows that $\mathcal{T}_{\text{subspace}} \subseteq \mathcal{T}'$.

Discussion. The map i given above is continuous on $(A, \mathcal{T}_{\text{discrete}})$ and is not continuous on $(A, \mathcal{T}_{\text{indiscrete}})$. Thus, if we start with the discrete topology on A and coarsen it, at some point the map i will cease to be continuous. This problem shows that the subspace topology is the last topology with respect to which i is continuous.

6.5 Given two continuous functions $f, g : X \to Y$ where X is a topological space and Y is a Hausdorff space.

(a) Suppose $A \subseteq X$ and $f(a) = g(a)$ for all $a \in A$. Prove that $f(x) = g(x)$ for all $x \in \overline{A}$.

Suppose $f(x) \neq g(x)$ for some $x \in \overline{A}$. Since the codomain is Hausdorff, we can find disjoint open sets $\mathcal{O}_f, \mathcal{O}_g$ containing $f(x), g(x)$ respectively. Both $f^{-1}(\mathcal{O}_f)$ and $g^{-1}(\mathcal{O}_g)$ are open sets containing x by continuity of f and g, which implies their intersection U is also an open set containing x. Since $x \in \overline{A}$, U must also contain an $a \in A$. This implies

$$a \in f^{-1}(\mathcal{O}_f) \cap g^{-1}(\mathcal{O}_g),$$

whence $f(a) \in \mathcal{O}_f$ and $g(a) \in \mathcal{O}_g$. But $f(a) = g(a)$, contradicting the disjointness of $\mathcal{O}_f$ and $\mathcal{O}_g$.

(b) Prove that the set $\{x \in X : f(x) = g(x)\}$ is closed.

Let A be the set in question. By (a) we have $\overline{A} \subset A$, whence A must be closed.

6.6 Prove that the Sorgenfrey line is a Hausdorff space.

Given $\alpha \neq \beta$; WLOG, say $\alpha < \beta$. Then $[\alpha, \beta)$ is an open set containing α, $[\beta, \beta + 1)$ is an open set containing β, and these two sets have empty intersection.

6.7 Prove that $\mathbb{Z}_{\text{furstenberg}}$ is a Hausdorff space.

Given $m \neq n$. Pick any $d > |m - n|$, and consider the two open sets $m + d\mathbb{Z}$ and $n + d\mathbb{Z}$. It's clear that these contain m and n, respectively. I claim they are also disjoint. Pick $x \in (m + d\mathbb{Z}) \cap (n + d\mathbb{Z})$. Then $x = m + dk = n + d\ell$ for some integers $k \neq \ell$. But this implies

$$|m - n| = d|\ell - k| > |m - n|,$$

a contradiction.

6.8 Consider $\mathbb{R}^2$ with respect to the Zariski topology. Prove that any two nonempty open sets intersect. (In particular, this space isn't Hausdorff!)

I claim that any two nonempty basis elements intersect. This implies that any pair of nonempty open sets intersect.

Recall that in the Zariski topology, a basis element is the support of a polynomial $f \in \mathbb{R}[x, y]$. (In other words: given any $f(x, y)$ that's a finite sum of terms of the form $cx^m y^n$ with $c \in \mathbb{R}$ and m, n non-negative integers, we can form the set $\mathcal{B}_f := \{(x, y) \in \mathbb{R}^2 : f(x, y) \neq 0\}$. The collection of all such sets $\mathcal{B}_f$ forms a basis of the Zariski topology.) Observe that the identically zero polynomial $f \equiv 0$ corresponds to the basis element $\varnothing$. Thus our goal is to prove that if both f and g have non-empty support, then there exists a point in $\mathbb{R}^2$ where neither f nor g vanishes. In other words:

Proposition 1. *If $f, g \in \mathbb{R}[x, y]$ and $fg \equiv 0$, then either $f \equiv 0$ or $g \equiv 0$.*

This looks like an obvious statement, but it's a bit tricky to prove. One approach is to use the Fundamental Theorem of Algebra: there exists a line in the plane with only finitely many zeros of f and g, which means that there are tons of points on that line at which both f and g are nonzero. To prove the Proposition without relying on the Fundamental Theorem of Algebra, we have to work a bit harder, however. We warm up with an easier version of the proposition that holds for polynomials in a single variable:

Lemma 2. *If $f, g \in \mathbb{R}[x]$ and $fg \equiv 0$, then either $f \equiv 0$ or $g \equiv 0$.*

Proof. Given $f \in \mathbb{R}[x]$, either $|f(x)| \to \infty$ as $x \to \infty$ or f is constant. Thus if $fg \equiv 0$, then both f and g must be constant. This immediately implies that one of f or g must be identically zero. $\square$

We now adapt this argument to prove our Proposition.

Proof of Proposition. Given any polynomial $f \in \mathbb{R}[x, y]$, write it in the form

$$f(x, y) = f_0(x) + f_1(x)y + \cdots + f_m(x)y^m$$

where $f_k \in \mathbb{R}[x]$ for all k and $f_m \not\equiv 0$. Similarly, write

$$g(x, y) = g_0(x) + g_1(x)y + \cdots + g_n(x)y^n$$

where $g_k \in \mathbb{R}[x]$ for all k and $g_n \not\equiv 0$. Then $fg(x, y) = f_0(x)g_0(x) + \cdots + f_m g_n(x)y^{m+n}$. By our Lemma, $f_m g_n \not\equiv 0$, so either there exists $a \in \mathbb{R}$ such that $|fg(a, y)| \to \infty$ as $y \to \infty$ or $fg(x, y) = f_0(x)g_0(x)$. It follows that

$$fg \equiv 0 \implies fg \in \mathbb{R}[x].$$

Thus we may apply our Lemma to fg, which concludes the proof. $\square$

Discussion. We say a commutative ring has *zero divisors* if there exist ring elements $f, g \neq 0$ such that $fg = 0$. For example, the ring $\mathbb{R}$ has no zero divisors, while the ring $\mathbb{Z}/4\mathbb{Z}$ does (since $2 \times 2 \equiv 0$). In our proof above we showed that $\mathbb{R}[x]$ has no zero divisors; this crucially relied on knowing that $\mathbb{R}$ has no zero divisors. Then we proved that $\mathbb{R}[x, y]$ has no zero divisors, by viewing the ring in the form $(\mathbb{R}[x])[y]$ and using the non-existence of zero divisors in $\mathbb{R}[x]$. It turns out these two steps are really the same step: if $\mathcal{R}$ is a commutative ring with no zero divisors, then $\mathcal{R}[x]$ is a commutative ring with no zero divisors.

6.9 In class we saw an example of an open map that wasn't a closed map (the projection map from $\mathbb{R}^2_{\text{usual}}$ onto $\mathbb{R}_{\text{usual}}$). Can you find an example of a closed map that isn't an open map?

There are many examples. Perhaps the easiest is a constant map $\mathbb{R}_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ that sends $x \mapsto 0$, say. This is clearly a closed map, since the image of *any* set is closed. It's also not an open map, since the image of any set is not open.

6.10 (Challenge problem, optional!) Construct an open map $f : \mathbb{R}_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ that's not continuous at *any* point of $\mathbb{R}$.

I leave this to you to think about!