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**Williams College**
**Department of Mathematics and Statistics**

# MATH 402 : MEASURE THEORY

## Final Exam

**Wednesday, December 16th—Sunday, December 20th**

Time slots will shared soon. If you would prefer a time outside this date range, or cannot make any of the remaining time slots, please contact Leo directly to discuss.

## INSTRUCTIONS

The final exam will consist of three questions, to be discussed orally with me. The duration of the exam will be approximately 30 minutes; it will take place over Zoom. I will share a blackboard on the screen, and will act as a scribe as you describe your ideas, but apart from this I request that you not use any other aids (notes, books, paper, etc) during your exam.

The exam will begin with Question A, which will be asked of every student. The other two questions will be selected by coin flip from Lists B and C, one question from each. (See next page for questions.)

In all your responses, I request that you first explain the proof in the form of a high-level overview: aim for fewer than five sentences. However, you should be able to also derive everything that comes up in your proof from first principles, and I reserve the right to follow up on anything you mentioned. For example, if you mention that a function is measurable, I might ask you to define what it means for a function to be measurable; if you mention that the measure of any set can be well-approximated by the measure of a closed subset, I might ask you to prove that. In short, as you study the material, I want you to continually ask yourself the question: *can I define / prove this without looking it up*?

Often, it is during an exam that you realize for the first time that you don't properly understand something. This is not only natural, it is totally OK; I will give you as many hints as you need to get back on track. Although part of the exam is to see how far you can go on your own, the more valuable aspect of an oral exam is that it's a chance for some individualized learning.

**I strongly encourage you to practice for the exam with other folks from the class;** take turns in the role of the examiner. Don't go easy on your partner! Anytime something is unclear, follow up on it; anytime they use a theorem or a definition, ask them for more details.

Best of luck!

Leo

*Questions on next page...*

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## Final Exam

**QUESTIONS**

<u>**Question A**</u> (Big Picture)
Given a measurable non-negative function $f : [0, 1] \to \mathbb{R}$, give a (non-rigorous) derivation of a formula expressing the Lebesgue integral of $f$ in terms of a Riemann integral over measures of certain sets. Illustrate that your formula works by applying it to the characteristic function of $\mathbb{Q} \cap [0, 1]$.

<u>**List B**</u> (Measure Theory)

- **B.0** Construct a Vitali set, and prove that it's non-measurable. Prove that its exterior measure is positive.
- **B.1** Prove that any measure on $\mathbb{R}$ that's simultaneously translation invariant and satisfies countable additivity must be trivial (in that it assigns every interval the same measure).
- **B.2** Prove that exterior measure satisfies countable subadditivity.
- **B.3** Prove that Lebesgue measure satisfies countable additivity.
- **B.4** State and prove the Monotone Convergence Theorem for Lebesgue measurable sets. Give an example demonstrating why we must impose an extra hypothesis for the descending version.
- **B.5** Littlewood's first principle states: any set of finite measure is pretty much a finite union of cubes. State and prove a rigorous version of this.
- **B.6** Littlewood's second principle states: any measurable function is pretty much continuous. State and prove a rigorous version of this.
- **B.7** Littlewood's third principle states: if a sequence of measurable function $\{f_n\}$ converges pointwise to $f$, then the convergence is pretty much uniform. State and prove a rigorous version of this.

<u>**List C**</u> (Integration theory)

- **C.0** State and prove the Monotone Convergence Theorem for the Lebesgue integral. Give an example that demonstrates why we didn't state a version of the MCT for a monotonically decreasing sequence of functions.
- **C.1** Prove that any measurable function is the pointwise almost everywhere limit of **step** functions.
- **C.2** Given a non-negative measurable function $f$ that's bounded and supported on a set of finite measure. Prove that if the Lebesgue integral of $f$ is 0, then $f = 0$ almost everywhere.
- **C.3** State and prove the Bounded Convergence Theorem.
- **C.4** Prove that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then it's Lebesgue integrable, and that the two integrals agree.
- **C.5** State and prove Fatou's Lemma. (The version from the problem set, which is slightly different from the one in the book.)
- **C.6** State and prove the Dominated Convergence Theorem.
- **C.7** State and prove the Riesz-Fischer theorem for $L^1$.