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Last Time

"Proved" that $\ell(\mathbb{Q} \cap [0,1]) = 0$.

To formalize this, write down properties ~~the~~ any reasonable notion of length should have.

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(0) length is a function $\ell : \mathcal{P}(\mathbb{R}) \to [0,\infty]$.

(1) $\forall A, B$ disjoint, $\ell(A \cup B) = \ell(A) + \ell(B)$

(2) $\forall A, B$, $\ell(A \cup B) \leq \ell(A) + \ell(B)$

$\ell([0,1] \cup [5,6]) \neq 6$?

$\ell([0,2] \cup [5,6])$

Maybe "measure" is better word than length.

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~~(3)~~ If $A \subseteq B$, then $\ell(A) \leq \ell(B)$.

Relationships between different properties.

$(2) \Rightarrow (3)$? HW

$\forall A, B$, $\ell(A \cup B) = \ell(A) + \ell(B) - \ell(A \cap B)$

$(1) \Rightarrow (2)$ HW

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(2') $\exists A \subseteq \mathbb{R}$ s.t. $\ell(A) > 0$

(3') $\ell(A + \lambda) = \ell(A)$ $\forall \lambda \in \mathbb{R}$.

"translation invariant"

(4') $\ell((0,1)) = 1$

(5') $\ell(\phi) = 0$, $\ell(\mathbb{R}) = \infty$

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We know $\ell([a,b])$ should be $b-a$...

so need something of the flavor (4').

(6') Should be able to scale length.

(7') ℓ is surjective

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~~Ben~~ $(4') \Rightarrow (2')$

$(7') \Rightarrow (2')$

~~Next~~

Stephen questions (6')...

$(1) \Rightarrow (5')$ ✓ Ben doesn't.

—— – x ——

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Pare down to

(0) $\ell : \mathcal{P}(\mathbb{R}) \to [0,\infty]$

(1) $\ell(A \cup B) = \ell(A) + \ell(B)$ $\forall A, B$ disjoint

(2'') $\ell([a,b]) = b-a$

$\ell((0,1))$

$\ell(\{3/2\}) = \ell([3/2, 3/2])$
$= 0$. by (2'')

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$\ell((0,1)) + \ell(0) + \ell(1) = \ell([0,1])$

$\ell([0,0]) = 0 = \ell([1,1])$

$\Rightarrow \ell((0,1)) + 0 + 0 = 1$ ✓

Recap of proof that $\ell(\mathbb{Q} \cap [0,1]) = 0$.

Write $\mathbb{Q} \cap [0,1] = \{q_1, q_2, q_3, \dots\}$

Pick interval $I_1 \ni q_1$ w/ $\ell(I_1) = \frac{\varepsilon}{4}$

Pick interval $I_2 \ni q_2$ w/ $\ell(I_2) = \frac{\varepsilon}{8}$

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$\forall k$, pick interval $I_k \ni q_k$ w/ $\ell(I_k) = \frac{\varepsilon}{2^{k+1}}$

Then $\mathbb{Q} \cap [0,1] \subseteq \bigcup_{k=1}^{\infty} I_k$

$\Rightarrow \ell(\mathbb{Q} \cap [0,1]) \le \ell\left(\bigcup_{k=1}^{\infty} I_k\right)$

$\le \sum_{k=1}^{\infty} \ell(I_k)$

$= \frac{\varepsilon}{2} < \varepsilon.$ ■

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Lemma: If $A \subseteq B$, then $\ell(A) \le \ell(B)$.

Proved on HW.

We need another lemma:

Lemma: $\ell\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \ell(A_k)$

We already know this for two sets:

$\ell(A_1 \cup A_2) \le \ell(A_1) + \ell(A_2)$

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Adam points out ~~ask~~ an easier proof: cover $\mathbb{Q} \cap [0,1]$ by

$\bigcup_{k=1}^{\infty} [q_k, q_k]$.

...

To approach countable subadditivity lemma, can we prove

$\ell(A_1 \cup A_2 \cup A_3) \le$
$\le \ell(A_1 \cup A_2) + \ell(A_3)$

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$\le \ell(A_1) + \ell(A_2) + \ell(A_3)$

By induction, we deduce

$\ell\left(\bigcup_{k=1}^{N} A_k\right) \le \sum_{k=1}^{N} \ell(A_k)$

Recall:

$\sum_{k=1}^{\infty} \ell(A_k) = \lim_{N \to \infty} \sum_{k=1}^{N} \ell(A_k)$

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We know

$$\ell\left(\bigcup_{k=1}^{N} A_k\right) \le \sum_{k=1}^{N} \ell(A_k)$$

And

$$\sum_{k=1}^{\infty} \ell(A_k) = \lim_{N\to\infty} \sum_{k=1}^{N} \ell(A_k)$$

<u>Want</u>:

$$\ell\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \ell(A_k)$$

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$$\ell\left(\lim_{N\to\infty} \bigcup_{k=1}^{N} A_k\right)$$

$$= \lim_{N\to\infty} \underbrace{\ell\left(\bigcup_{k=1}^{N} A_k\right)}_{\le \sum_{k=1}^{N} \ell(A_k)}$$

$$\le \lim_{N\to\infty} \sum_{k=1}^{N} \ell(A_k)$$

$$= \sum_{k=1}^{\infty} \ell(A_k)$$

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Thus, if

$$\ell\left(\lim_{N\to\infty} \bigcup_{k=1}^{N} A_k\right) = \lim_{N\to\infty} \ell\left(\bigcup_{k=1}^{N} A_k\right)$$

then we can prove countable subadditivity.

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