1

Last time:

S'pose $\ell : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ satisfies

(1) $\ell(A \cup B) = \ell(A) + \ell(B)$ $\forall A, B$ disjoint

and (2) $\ell([a,b]) = b-a$.

Conj: For any ctbly infinite collection

2

of sets $A_k$,

$$\ell\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \ell(A_k).$$

This implies $\ell(\mathbb{Q} \cap [0,1]) = 0$

More generally, the Countable Subadditivity Conjecture implies $\ell(S) = 0$ for any countable $S \subseteq \mathbb{R}$.

3

An example of comput. length of an uncountable set: the Cantor set.

(discovered by Henry Smith)

$C_0$ : $[0,1]$

$C_1$ : $[0,\frac{1}{3}] \cup [\frac{2}{3},1]$

$C_2$ : $[0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup$

4

$\cup [\frac{8}{9},1]$.

$\vdots$

The Cantor set is

$$C := \bigcap_{k=1}^{\infty} C_k$$

$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

$C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup$
$\cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$

$\vdots$

5

$C$ is non-$\emptyset$:

$0 \in C$

Any endpt. of one of the intervals appearing in $C_n$ is an element of $C$.

Thus $C$ is @ least countably infinite.

6

Turns out,

$C$ is uncountable

and $C$ is totally disconnected

(i.e. $C$ contains no non-$\emptyset$ open intervals)

Note: $\ell(C) = 0$

$\ell(C_n) = \left(\frac{2}{3}\right)^n$ and $C \subseteq C_n$ $\forall n$.

**1**

Last time, we proved that the Countable Subadditivity Conj. follows from

$$(*)\quad \ell\left(\lim_{N\to\infty} \bigcup_{k=1}^{N} A_k\right) = \lim_{N\to\infty} \ell\left(\bigcup_{k=1}^{N} A_k\right)$$

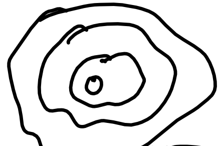
**2**

A cleaner way to write $(*)$ is

Conjecture: Given,

$$B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$$

$$\ell\left(\lim_{n\to\infty} B_n\right) = \lim_{n\to\infty} \ell(B_n)$$



$$\lim_{n\to\infty} B_n := \bigcup_{n=1}^{\infty} B_n$$

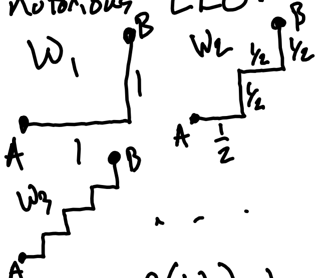
**3**

This type of Statement is called a Limit Exchange Operation.

LEOs are hard, and annoying, and will important in this class.

**4**

Example of a notorious LEO:




$$\lim_{n\to\infty} \ell(W_n) = \lim_{n\to\infty} 2 = 2$$

$$\ell\left(\lim_{n\to\infty} W_n\right) = \sqrt{2}.$$

**5**

In 1st class, Ben proposed the following notion of length: given $E \subseteq \mathbb{R}$,

$$m_*(E) := \inf \sum_{k=1}^{\infty} |I_k|$$

over all possible countable coverings of E by closed intervals $I_k$

**6**

i.e. given E, find a countable set of closed intervals $I_k$ s.t.

$$E \subseteq \bigcup_{k=1}^{\infty} I_k$$

$m_*(E) =$ inf over all these of $\sum_{k=1}^{\infty} |I_k|$.

**1**

It's clear that $m_*(E) \in [0, \infty]$.

Is it true that ~~$m_*(E)$~~

$m_*([a,b]) = b-a$?

__Observe__: $m_*([a,b]) \le b-a$ b/c $[a,b]$ is a cover of $[a,b]$.

**2**

Want: ~~$m_*(E[a,b])$~~

$m_*([a,b]) \ge b-a$.

$|[a,b]| := b-a$

$m_*(E) := \inf \sum_{k=1}^{\infty} |I_k|$

Given an arbitrary cover $[a,b] \subseteq \bigcup_{k=1}^{\infty} I_k$

**3**

Want:

$b-a \le \sum_{k=1}^{\infty} |I_k|$

__Problem__: $\sum_{k=1}^{\infty} |I_k| =$

$= \lim_{N\to\infty} \sum_{k=1}^{N} I_k$

but $[a,b]$ might not be covered by $\bigcup_{k=1}^{N} I_k$.

**4**

__Compactness__

Dirichlet proof

__Propn__: If $f$ is cts. on $[a,b]$, then $f$ is bounded on $[a,b]$.

__Dirichlet's proof__:

Since $f$ is cts. on $[a,b]$, $\forall x \in [a,b]$, $f$ is bdd on

**5**

some neighborhood $I_x \ni x$.

And $[a,b] \subseteq \bigcup_{x\in[a,b]} I_x$

$f$ is bdd on each $I_x$, but there are __$\infty$ly__ __many__ $I_x$'s.

**6**

Dirichlet proof

$\exists \{A_1, A_2, \dots, A_n\} \subseteq$ finite

$\{I_x : x \in [a,b]\}$

s.t. $[a,b] \subseteq \bigcup_{k=1}^{n} A_k$.

Since $f$ is bdd on each $A_k$, and there are finitely many, $f$ is bdd on $[a,b]$. ∎