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Last class, studied

$$m_*(A) := \inf\left\{\sum_{k=1}^{\infty}|I_k| : \{I_k\} \text{ are a ctble set of closed intervals that cover } A\right\}$$

$$|[a,b]| := b-a.$$

$\{I_k\}$ is a cover of A iff

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$$A \subseteq \bigcup_{k=1}^{\infty} I_k.$$

m_* is "exterior measure".

<u>Prop$^{\underline{n}}$</u>: $m_*([a,b]) = b-a$

<u>Pf</u>: Since $[a,b]$ covers itself, $m_*([a,b]) \le b-a$.

Pick any ctble cover of $[a,b]$ by closed intervals $\{I_k\}$.

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We have $[a,b] \subseteq \bigcup_{k=1}^{\infty} I_k$

<u>Want</u>: to turn this <u>infinite</u> cover into a <u>finite</u> cover.

<u>Q</u>: is it true that every closed cover of $[a,b]$ has a finite subcover?

No: $[a,b] \subseteq \bigcup [x,x]$ over $x \in [a,b]$.

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How to get from closed to open cover?

(1) Remove endpts of each I_k.

(2) Extend I_n slightly in both directions to get a slightly larger open interval.

$\forall k$, pick O_k an open interval s.t. $O_k \supseteq I_k$ and $|\overline{O_k}| \le (1+\varepsilon)|I_k|$

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We have

$$[a,b] \subseteq \bigcup_{k=1}^{\infty} I_k \subseteq \bigcup_{k=1}^{\infty} O_k$$

Since $[a,b]$ is closed+bdd, Heine-Borel $\Rightarrow$ $[a,b]$ is compact, i.e. $\exists$ some <u>finite</u> set

$$\{O_{k_1}, O_{k_2}, \dots, O_{k_n}\} \subseteq \{O_k\}$$

s.t. $[a,b] \subseteq \overline{O_{k_1}} \cup \cdots \cup \overline{O_{k_n}}$

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Thus,

$$b-a \le |\overline{O_{k_1}}| + |\overline{O_{k_2}}| + \cdots + |\overline{O_{k_n}}|$$
$$\le (1+\varepsilon)|I_{k_1}| + (1+\varepsilon)|I_{k_2}| + \cdots + (1+\varepsilon)|I_{k_n}|$$
$$\le (1+\varepsilon)\sum_{k=1}^{\infty}|I_k|$$

Since $\varepsilon > 0$ is arbitrary,

$$\Rightarrow b-a \le \sum_{k=1}^{\infty}|I_k|$$

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So, $b-a$ is a lower bd on

$$\left\{ \sum_{k=1}^{\infty} |I_k| : \{I_k\} = \text{closed int. cover of } [a,b] \right\}$$

$\Rightarrow b-a \le m_*([a,b])$. ∎

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Nice Props of m_*

① $A \subseteq B \Rightarrow m_*(A) \le m_*(B)$

Pf: Any cover of B covers A. ∎

Corollary: If A is bdd, then $m_*(A) = \text{finite}$.

② Countable Subadditivity

$$m_*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} m_*(A_k)$$

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Strategy:

~~S'pose~~

Cover A_1 by intervals $I_k^{(1)}$

s.t. $m_*(A_1) = \sum_{k=1}^{\infty} |I_k^{(1)}|$.

Cover A_2 by Intervals $I_k^{(2)}$

s.t. $m_*(A_2) = \sum_{k=1}^{\infty} |I_k^{(2)}|$

⋮

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Then $\{I_k^{(j)}\}$ would cover $\bigcup_{k=1}^{\infty} A_k$.

$$\Rightarrow m_*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{j,k} |I_k^{(j)}|$$

$$= \sum_{j} m_*(A_k)$$

This doesn't work, but almost as good:

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Observe that

$$m_*(A_k) \le \sum_{j=1}^{\infty} |I_j^{(k)}|$$

for any cover of A_k

and $\exists$ cover $\{I_j^{(k)}\}$

s.t. $\sum_{j=1}^{\infty} |I_j^{(k)}| \le m_*(A_k) + \varepsilon$.

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Proof of ②:

If $m_*(A_j) = \infty$, done.

Else: Pick $\varepsilon > 0$.

$\exists$ cover $\{I_k^{(1)}\}$ of A_1

s.t. $\sum |I_k^{(1)}| \le m_*(A_1) + \frac{\varepsilon}{2}$

$\exists$ cover $\{I_k^{(2)}\}$ of A_2

s.t. $\sum |I_k^{(2)}| \le m_*(A_2) + \frac{\varepsilon}{4}$

⋮ $\qquad$ ⋮ $\qquad$ $\le m_*(A_3) + \frac{\varepsilon}{8}$

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Thus $\{I_k^{(j)}\}$ is a cover of $\bigcup_{k=1}^{\infty} A_k$

$$\Rightarrow m_*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{j,k} |I_k^{(j)}|$$

$$\le \sum_{k=1}^{\infty}\left(m_*(A_k) + \frac{\varepsilon}{2^k}\right)$$

$$= \sum_{k=1}^{\infty} m_*(A_k) + \varepsilon$$

■

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Q (Spencer):

Could we have used $(1+\varepsilon)$ factor instead of trick w/ $\frac{\varepsilon}{2^k}$?

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