1

Recall

$$m_*(A) := \inf \sum_{k=1}^{\infty} |I_k|$$

Note: there may not exist any cover $\{I_k\}$ of A s.t.

$$m_*(A) = \sum_{k=1}^{\infty} |I_k|$$

However: for any cover,

$$m_*(A) \le \sum_{k=1}^{\infty} |I_k|$$

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Also, $\forall \varepsilon > 0$, $\exists$ cover $\{I_k\}$ s.t.

$$m_*(A) + \varepsilon \ge \sum_{k=1}^{\infty} |I_k|$$

We proved:

Prop^n: $m_*([a,b]) = b-a$.

① Monotonicity: $A \subseteq B \Rightarrow m_*(A) \le m_*(B)$

② Countable subadditivity:

$$m_*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} m_*(A_j).$$

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③ $$m_*(A) = \inf_{\substack{\text{open} \\ O \supseteq A}} m_*(O).$$

Refresher on topology of $\mathbb{R}^n$

Q: What does it mean for $A \subseteq \mathbb{R}^n$ to be open?

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e.g.

is open b/c it's missing its boundary.

Def^n (informal):

~~x is in~~ $x \in \partial A$ (the boundary of A) iff no matter how close you zoom in on x, you'll still see

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~~pts from A and~~ ~~pts from~~ A^c.

"complement of A".

pts from interior of A and pts. from exterior of A.

Def^n (informal):

$x \in \text{int}(A)$ ("interior of A") iff once zoom in far enough, only see

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pts in A.

$x \in \text{ext}(A)$ ("exterior of A") iff when zoom in close enough on x, only see pts from A^c.

To make these rigorous: need ~~[illegible]~~ to understand what "zoom in" means.

1

Def$^{\underline{n}}$: Let

$$B_r(x) := \{y \in \mathbb{R}^n : |x-y| < r\}$$

(open) "ball of radius r around x".

e.g. ~~[illegible]~~ $B_0(x) =$ ~~[illegible]~~ $\emptyset$

$B_1(x)$ = ~~sphere~~ everything strictly inside sphere of radius 1 centered @ x.

2

Def$^{\underline{n}}$ (formal):

$x \in \mathrm{int}(A)$ <u>iff</u> $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq A$.

$x \in \mathrm{ext}(A)$ <u>iff</u> $x \in \mathrm{int}(A^c)$.

$x \in \partial A$ <u>iff</u> $x \notin \mathrm{int}(A) \cup \mathrm{ext}(A)$.

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Def$^{\underline{n}}$: A is <u>open</u> iff $A = \mathrm{int}(A)$.

A is <u>closed</u> iff $\partial A \subseteq A$.

Properties:

(i) A is closed <u>iff</u> A^c is open.

(ii) $\emptyset$ and $\mathbb{R}^n$ are clopen (both open <u>and</u> closed)

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(iii) A is closed <u>iff</u> it contains all its limit pts.

(iv) $\bigcup_\alpha O_\alpha$ is open if all O_α are open and $\bigcap_{j=1}^{N} O_j$ is open if all O_j are open.

<u>Caution</u>: infinite intersect

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of opens might <u>not</u> be open,

e.g. $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\} = [0,0]$

Heine-Borel Thm:

Given $A \subseteq \mathbb{R}^n$. The following are equivalent:

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(i) A is closed and bounded.

(ii) Every sequence in A has a convergent subsequence.

(iii) A is compact, i.e. every cover of A by open sets contains a finite subcover of A.

1

(3) $m_*(A) = \inf\limits_{\substack{\text{open}\\ O \supseteq A}} m_*(O)$.

Proof idea:

(1) $\Rightarrow m_*(A) \le m_*(O)$ $\forall O \supseteq A$

$\Rightarrow m_*(A) \le \inf m_*(O)$.

S'pose we found a perfect cover of A by closed intervals,

2

s.t. $\{I_k\}$ is a cover of A s.t.

$$m_*(A) = \sum_{k=1}^{\infty} |I_k|.$$

If we can find open O s.t. $O \supseteq \bigcup I_k$, and

$m_*(O) \le m_*(\bigcup I_k) + \varepsilon \le \sum m_*(I_k) + \varepsilon =$

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$= \sum |I_k| + \varepsilon = m_*(A) + \varepsilon$

Pf: First, $m_*(A) \le \inf\limits_{\substack{O \text{ open}\\ \text{s.t. } O \supseteq A}} m_*(O)$

ETS. $\forall \varepsilon > 0$, $\exists O \supseteq A$ open s.t. $m_*(O) \le m_*(A) + \varepsilon$.

Pick $\varepsilon > 0$. $\exists$ closed interval cover $\{I_j\}$ of A s.t. $m_*(A) + \varepsilon \ge \sum |I_j|$.

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$\exists$ open interval $O_1 \supseteq I_1$ s.t. $|O_1| \le |I_1| + \varepsilon$

$\exists$ open interval $O_2 \supseteq I_2$ s.t. $|O_2| \le |I_2| + \varepsilon/2$

$\vdots$

$$\Rightarrow m_*\left(\bigcup_{j=1}^{\infty} O_j\right) \le \sum_{j=1}^{\infty} m_*(O_j) \le \sum_{j=1}^{\infty}\left(|I_j| + \frac{\varepsilon}{2^{j-1}}\right) = \sum |I_j| + 2\varepsilon$$

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$\le m_*(A) + \varepsilon + 2\varepsilon$

So

$$m_*(\underbrace{\bigcup O_j}_{\text{open}}) \le m_*(A) + 3\varepsilon.$$

So $\exists$ open set $(\bigcup O_j)$ s.t. $m_*(O) \le m_*(A) + 3\varepsilon$.

$\Rightarrow \inf\limits_{O \supseteq A} m_*(O) \le m_*(A) + 3\varepsilon$. ∎

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