1

In HW,

$$\mu(A) := \lim_{N\to\infty} \frac{1}{N} \#\{A \cap \tfrac{1}{N}\mathbb{Z}\}$$

Why is this natural?

"Measuring rationalness of A".

This reduces potentially uncountable questions to countable ones (even finite!).

2

This also might be reasonable since $\mathbb{Q}$ is dense in $\mathbb{R}$.

So might hope to approx. measure just using $\mathbb{Q}$.

<u>And</u> this satisfies

$$\mu\left(\bigsqcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} \mu(I_n)$$

and $\mu([a,b]) = b-a$.

3

Can think about this probabilistically....

OR we can rewrite this as (assuming $A \subseteq [0,1]$)

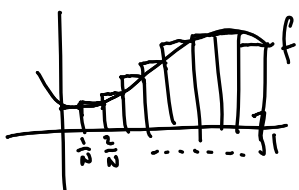
$$\mu(A) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \chi_A\left(\frac{k}{N}\right)$$

where $\chi_A(t) := \begin{cases} 1 & t \in A \\ 0 & t \notin A \end{cases}$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{N} \chi_A\left(\frac{k}{N}\right)$$

4

This looks like a Riemann sum: recall



$$\int_0^1 f(t)\,dt = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{N} f\left(\frac{k}{N}\right)$$

Riemann integral

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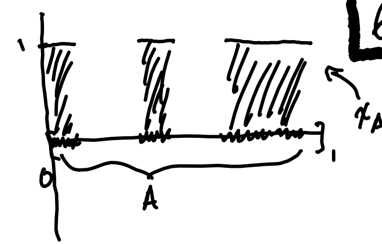
Thus, for $A \subseteq [0,1]$,

$$\mu(A) = \int_0^1 \chi_A(t)\,dt$$

* χ_A is the "characteristic f^n" of A.

$\int_0^1 \chi_A(t)\,dt$ is a natural way to think about length.

6



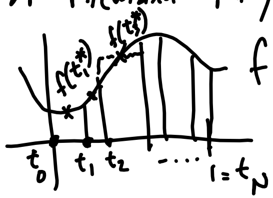
Turns out that

$\int_0^1 \chi_{\mathbb{Q}}(t)\,dt$ doesn't exist

b/c $\lim_{\nu\to\infty} \frac{1}{\nu} \sum \chi_A(\ldots)$ doesn't exist

1

More flexible notions of Riemann integrals:



$$\int_0^1 f = \inf \sum_{k=1}^{N} f(t_k^*)\,(t_k - t_{k-1})$$

Turns out for $f = \chi_A$ this is the J_*.

2

We've been exploring exterior measure m_*.

We've proved

(1) m_* is monotonic

$$A \subseteq B \Rightarrow m_*(A) \le m_*(B).$$

(2) Countable subadd.

$$m_*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} m_*(A_j)$$

(3) Approximation by opens

$$m_*(A) = \inf_{\substack{\text{open} \\ \mathcal{O} \supseteq A}} m_*(\mathcal{O}).$$

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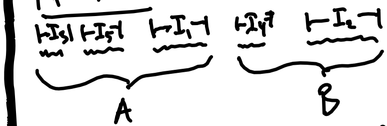
~~(4)~~ $m_*(A \cup B) = m_*(A) + m_*(B)$

if A, B disjoint.

Pf idea:



Start w/ perfect cover of $A \cup B$. Then partition into separate covers of A and B: $A = I_1 \cup I_3 \cup I_5$, $B = I_2 \cup I_4$

4

Then

$$m_*(A) + m_*(B) = |I_1| + |I_3| + |I_5| + |I_2| + |I_4|$$
$$= |I_1| + |I_2| + \dots + |I_5|$$
$$= m_*(A \cup B).$$

Potential issues:

(a) We need each I_k to intersect precisely one of A xor B.

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(b) Perfect cover of $A \cup B$ might not be finite... or even exist.

Resolution of (b):

Given $\varepsilon > 0$, $\exists \{I_k\}$ a closed interval cover of $A \cup B$ s.t.

$$m_*(A \cup B) + \varepsilon \ge \sum_{k=1}^{\infty} |I_k|$$

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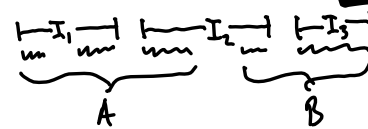
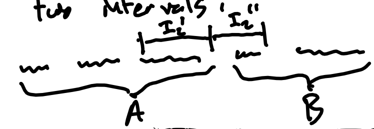
Resolution of (a):



Problem: I_2 intersects both A and B.

Break up I_2 into two intervals:



1

(4') $m_*(A \cup B) = m_*(A) + m_*(B)$

if $d(A,B) > 0$.

where $d(A,B) := \inf_{\substack{a \in A \\ b \in B}} |a-b|$

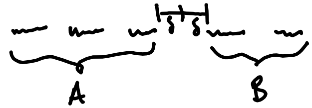
Pf: We know

$m_*(A \cup B) \leq m_*(A) + m_*(B)$

by (2). E.T.S. that

$m_*(A) + m_*(B) \leq m_*(A \cup B) + \varepsilon$

2



Let $\dfrac{d(A,B)}{2} =: \delta$

Note: any interval of length $\leq \delta$ can't intersect both A and B.

Pick $\varepsilon > 0$. $\exists \{I_k\}$ closed interval cover of $A \cup B$ s.t. $m_*(A \cup B) + \varepsilon \geq \sum_{k=1}^{\infty} |I_k|$

3

~~W~~ We may assume

- $\forall k$, $I_k \cap (A \cup B)$.
- $\forall k$, $|I_k| \leq \delta$.

Thus, $\forall k$, ~~__~~ either

$I_k \cap A \neq \emptyset$ xor

$I_k \cap B \neq \emptyset$.

Let $J := \{k \in \mathbb{N} : I_k \cap A \neq \emptyset\}$.

Then

4

$m_*(A) + m_*(B)$

$\leq \sum_{k \in J} |I_k| + \sum_{k \notin J} |I_k|$

$= \sum_{k=1}^{\infty} |I_k|$

$\leq m_*(A \cup B) + \varepsilon$. ∎

(5) If $A = \bigcup_{j=1}^{\infty} I_j$ where I_j are almost disjoint intervals,

then $m_*(A) = \sum_{j=1}^{\infty} |I_j|$.

5

Almost disjoint

means $\operatorname{int}(I_j) \cap \operatorname{int}(I_k) = \emptyset$ $\forall j \neq k$.

Pf idea:

$m_*(I_1 \cup I_2) \overset{?}{=} m_*(I_1) + m_*(I_2)$

(4') $\Rightarrow m_*(I_1' \cup I_2') = m_*(I_1') + m_*(I_2')$

6

Def^{n}: We say $A \subseteq \mathbb{R}^n$ is measurable iff $\forall \varepsilon > 0$

$\exists \mathcal{O} \supseteq A$ s.t. $m_*(\mathcal{O} \setminus A) \leq \varepsilon$.

open

In this case, we write

$m_*(A) = m(A)$.