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Last time:

Def$^{\underline{n}}$: $A \subseteq \mathbb{R}^n$ is <u>measurable</u> iff $\forall \varepsilon > 0$, $\exists$ open $\mathcal{O} \supseteq A$ s.t.

$$m_*(\mathcal{O} \setminus A) \leq \varepsilon.$$

$m_*(S)$ exists $\forall S \subseteq \mathbb{R}^n$.

The issue is how the measure of S combined

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w/other sets behaves.

e.g.

$\exists B \subseteq [0,1]$ s.t.

$m_*(B) = 1$ and

$m_*([0,1] \setminus B) = 1$

By contrast, measurable sets don't have such pathological behavior.

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Turns out every set you encounter in the wild is measurable: every non-measurable set <u>requires</u> the axiom of choice.

<u>Thm (Solovay, 1970):</u>

$\exists$ model of ZF in which every set is measurable.

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<u>Axiom of Choice:</u>

Given a non-empty collection of non-empty sets, their cartesian product is non-empty.

This is <u>true</u> for <u>finite</u> collections of sets.

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Consider $\mathbb{Q} \cap [0,1]$.

What $\mathcal{O}$ approximates this well?

$\mathcal{O} = (0,1)$? :(

(What's $m_*(\mathbb{Q} \cap [0,1])$? $= 0$)

Good choice for $\mathcal{O}$:

enumerate

$\mathbb{Q} \cap [0,1] = \{q_1, q_2, \dots\}$,

Cover q_k by $\left(q_k - \frac{\varepsilon}{2^{k+1}}, q_k + \frac{\varepsilon}{2^{k+1}}\right)$

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Then

$$\mathcal{O} := \bigcup_{k=1}^{\infty} \left(q_k - \frac{\varepsilon}{2^{k+1}}, q_k + \frac{\varepsilon}{2^{k+1}}\right)$$

is open, covers $\mathbb{Q} \cap [0,1]$, and $m_*(\mathcal{O}) \leq \varepsilon$.

$$\Rightarrow m_*(\mathcal{O} \setminus \mathbb{Q} \cap [0,1]) \leq m_*(\mathcal{O}) \leq \varepsilon.$$

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Nice properties of measurable sets:

(1) Open sets are measurable.

(1') Closed sets are measurable

(2) Any set of exterior measure 0 is measurable.

(3) Countable unions of measurable sets are measurable.

(3') Countable intersections

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of measurable sets are measurable.

(4) A msble $\Rightarrow$ A^c msble.

Moreover:

Thm: $m\left(\bigsqcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} m(A_j)$.

Pf idea: $\forall j$ $\exists$ "nice" F_j s.t. $m(F_j) \approx m(A_j)$

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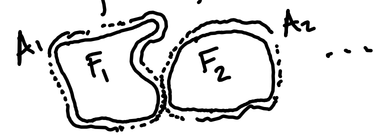
This would reduce problem to proving

$$m\left(\bigsqcup_{j=1}^{\infty} F_j\right) = \sum_{j=1}^{\infty} m(F_j).$$

To guarantee F_j's being disjoint, pick

$$F_j \subseteq A_j \quad \forall j.$$



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Immediately:

$$m\left(\bigsqcup A_j\right) \geq m\left(\bigsqcup F_j\right)$$

We know from last time that if $\bigsqcup F_j$ is a finite union, and if F_j's are separated from one another by positive distance, then

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$$m\left(\bigsqcup_{j=1}^{N} F_j\right) = \sum_{j=1}^{N} m(F_j)$$

$$\Rightarrow \qquad \geq \sum_{j=1}^{N} m(A_j) - \varepsilon$$

$$\Rightarrow m\left(\bigsqcup_{j=1}^{\infty} A_j\right) \geq \sum_{j=1}^{N} m(A_j) - \varepsilon$$

Taking $N \to \infty$, get

$$m\left(\bigsqcup_{j=1}^{\infty} A_j\right) \geq \sum_{j=1}^{\infty} m(A_j) - \varepsilon.$$

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$\varepsilon \to 0^+$

$$\Rightarrow m\left(\bigsqcup_{j=1}^{\infty} A_j\right) \geq \sum_{j=1}^{\infty} m(A_j)$$

And countable subadditivity

$$\Rightarrow m\left(\bigsqcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} m(A_j).$$

"QED"

Ingredients:

- Nice $F_j \subseteq A_j$ s.t. $m(F_j) \geq m(A_j) - \varepsilon$
- $d(F_j, F_k) > 0$ when $j \neq k$.

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Lemma: If K and K' are disjoint compact sets, then $d(K,K')>0$.

Proof of Lemma in book.

Thus want F_j's to be compact; i.e. (by Heine-Borel) want F_j's closed and bdd.

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If A_j's were bdd, we just choose closed $F_j \subseteq A_j$, done!

Note: Even a measure 0 set can be unbdd! e.g. $\mathbb{Q}$.

We're ready to prove:

Pf: Given A_j disjoint, measurable.

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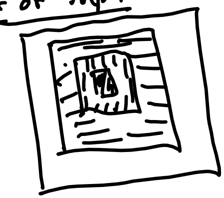
Step 1: Given any $A \subseteq \mathbb{R}^n$, we can express it as a countable disjoint union of bdd sets.

Pf of Step 1:



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Pf of Step 1:

Pick any sequence $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$ of cubes in $\mathbb{R}^n$ whose union is $\mathbb{R}$.

Set $C_1' := C_1$

$C_2' := C_2 \setminus C_1$

$C_3' := C_3 \setminus C_2$

$\vdots$

Note $\bigsqcup_{j=1}^{\infty} C_j' = \mathbb{R}^n$.

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Then $\bigsqcup_{k=1}^{\infty} (A \cap C_k') = A$

and $A \cap C_k'$ is bdd. //

By Step 1, we may assume all A_j's are bdd.

Step 2: $\forall j$, $\exists$ closed $F_j \subseteq A_j$ s.t. $m(A_j \setminus F_j) \le \varepsilon$

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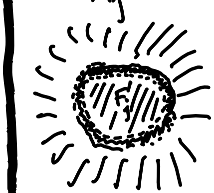
Pf. of Step 2:

A_j is msble

$\Downarrow$

A_j^c is msble

$\Rightarrow \exists$ open $\mathcal{O} \supseteq A_j^c$ s.t. $m_*(\mathcal{O} \setminus A_j^c) \le \varepsilon$



Let $F_j := \{\mathcal{O}^c\} \Rightarrow$ $F_j \subseteq A_j$ and $m(A_j \setminus F_j) \le \varepsilon$. //

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Step 3: Win.

$F_j \subseteq A_j \Rightarrow F_j$ bdd.

and F_j closed

$\Rightarrow F_j$ compact.

And $F_j \subseteq A_j \Rightarrow F_j$'s all pairwise disjoint.

Lemma $\Rightarrow d(F_j, F_k) > 0$ whenever $j \neq k$

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So:

$$m\left(\bigsqcup_{j=1}^{\infty} A_j\right) \geq m\left(\bigsqcup_{j=1}^{N} F_j\right)$$

$$= \sum_{j=1}^{N} m(F_j)$$

$$\geq \sum_{j=1}^{N}\left(m(A_j) - \frac{\varepsilon}{2^j}\right)$$

$$\geq \sum_{j=1}^{N} m(A_j) - \varepsilon$$

$N \to \infty \Rightarrow m\left(\bigsqcup_{j=1}^{\infty} A_j\right) \geq \sum_{j=1}^{\infty} m(A_j) - \varepsilon$

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Take $\varepsilon \to 0^+$

$$\Rightarrow m\left(\bigsqcup_{j=1}^{\infty} A_j\right) \geq \sum_{j=1}^{\infty} m(A_j)$$

Ctble Additivity

$$\Rightarrow m\left(\bigsqcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} m(A_j).$$

Q.E.F.D.

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