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~~Last time:~~ we proved

Thm: (countable additivity):

~~If $\{A_k\}$ is a collection of msble sets~~

~~then~~ $m\left(\bigsqcup_{j=1}^{\infty} A_k\right) = \sum_{j=1}^{\infty} m(A_k)$

This property + other nice properties of msble sets gives rise to a

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general strategy for evaluating / bounding measures of sets:

given an arbitrary $S \subseteq \mathbb{R}^n$, approximate S by a nice msble set, e.g. open. Then use properties of msble sets!

Recall $\forall S \subseteq \mathbb{R}^n$, $\exists \underset{\text{open}}{O} \supseteq S$ s.t. $m_*(O) \leq m_*(S) + \varepsilon$.

Note: $m_*(S) \leq m_*(O)$

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Note: if S has $m_*(S) = \infty$, and S is msble,

$\exists \underset{\text{open}}{O} \supseteq S$ s.t. $m_*(O \setminus S) \leq \varepsilon$.

This strategy plays a role in solution to 3.4:

Thm: $A \subseteq \mathbb{R}^n$ is msble iff $\forall S \subseteq \mathbb{R}^n$, $m_*(S) = m_*(S \cap A) + m_*(S \cap A^c)$

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This is called "Carathéodory Criterion."

Our def^n of measurable was given by Lebesgue. (often we say A is "Lebesgue measurable")

Advantage of Lebesgue's def^n:

More intuitive.

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But Lebesgue's def^n uses the concept of open set.

Advantage of Carathéodory's def^n

Doesn't rely on any notion of topology

hence generalizes.

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An important consequence of ctble additivity:

Monotone Convergence Thm:

"Any monotone sequence of msble sets A_k that converge to A satisfy $m(A) = \lim_{k\to\infty} m(A_k)$"

Def^n: A sequence A_k is "monotonically increasing" iff $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$

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A_k is "monotonically decreasing" iff

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$$

If A_k is monotone. increasing, it converges to $\bigcup_{k=1}^{\infty} A_k$

& We write $A_k \nearrow A$, where $A := \bigcup_{k=1}^{\infty} A_k$.

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If A_k mon. decr., we write

$$A_k \searrow A$$

where $A := \bigcap_{k=1}^{\infty} A_k$.

Formal statement of MCT:

~~(i)~~ Given $A_k \subseteq \mathbb{R}^n$ all msble.

~~(i) $A_k \nearrow A \Rightarrow m(A) = \lim_{k\to\infty} m(A_k)$~~

~~(ii) $A_k \searrow A \Rightarrow \dots$~~

(i) $A_k \nearrow A \Rightarrow m(A) = \lim_{k\to\infty} m(A_k)$

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(ii) $A_k \searrow A \Rightarrow m(A) = \lim_{k\to\infty} m(A_k)$

Provided $m(A_k) < \infty$ for some k.

This is an example of a LEO that works well.

Note: the condition $m(A_k) < \infty$ in (ii) is necessary.

e.g. $A_n \cancel{:= (n, n+\frac{1}{n})}$

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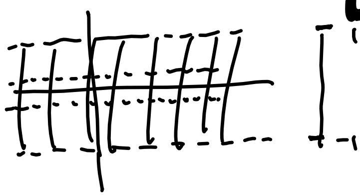
In $\mathbb{R}^2$:



$A_1 = \mathbb{R} \times [-1, 1]$

$A_2 = \mathbb{R} \times [-\frac{1}{2}, \frac{1}{2}]$

$\vdots$

$A_n = \mathbb{R} \times [-\frac{1}{n}, \frac{1}{n}]$

$A_n \searrow \mathbb{R}\times\{0\}$, but $m(A_k) = \infty\ \forall k$, $m(\mathbb{R}\times\{0\}) = 0$.

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In $\mathbb{R}$:

$A_n = \mathbb{R} \setminus (-n, n)$

$A_n \searrow \emptyset$

but $m(A_n) = \infty$

$m(\emptyset) = 0$.

Pf: (i) We're trying to prove $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$.

LHS looks like ctble add, but A_k's not disjoint!

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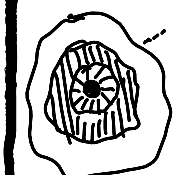
To make them disjoint,



Let

$A_2' := A_2 \setminus A_1$

$A_3' := A_3 \setminus A_2$

$\vdots$

$A_n' := A_n \setminus A_{n-1}$ $\forall n \geq 2$

and $A_1' := A_1$.

Then:

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Then:

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigsqcup_{k=1}^{\infty} A_k'\right)$$

$$= \sum_{k=1}^{\infty} m(A_k')$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} m(A_k')$$

$$= \lim_{N\to\infty} m\left(\bigsqcup_{k=1}^{N} A_k'\right)$$

$$= \lim_{N\to\infty} m(A_N). \quad \blacksquare$$

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(ii) proved in book.

We've shown that any msble set can be well-approx by an open set and by a closed set.

Turns out we can approximate by even nicer sets.

~~Prop^{n}:~~

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Prop^{n}: Given $m(A) < \infty$. Then

(1) $\exists$ compact $K \subseteq A$ s.t. $m(A \setminus K) \leq \varepsilon$.

(2) $\exists$ a finite union of closed cubes $\bigcup_{j=1}^{N} C_j$ s.t. $m\left(A \,\triangle\, \bigcup_{j=1}^{N} C_j\right) \leq \varepsilon$.

Here, $S \triangle T := (S \setminus T) \cup (T \setminus S)$.

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$S \triangle T$ is the "symmetric difference" of S, T:

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(2)

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We'll prove (2): Pick any $\varepsilon > 0$.

$\exists$ covering $\{C_j\}$ of A by ctbly many closed cubes s.t.

$$A \subseteq \bigcup_{j=1}^{\infty} C_j \quad \text{and}$$

$$\sum_{j=1}^{\infty} |C_j| \leq m(A) + \varepsilon.$$

$$m(A) < \infty \Rightarrow \sum_{j=1}^{\infty} |C_j| < \infty$$

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Thus $\exists N$ s.t.

$$\sum_{j>N} |C_j| \leq \varepsilon.$$

$$m\left(A \,\triangle \bigcup_{j\leq N} C_j\right) =$$

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$$m\Big(A \,\triangle \bigcup_{j\le N} C_j\Big) =$$

$$= m\Big(A \setminus \bigcup_{j\le N} C_j\Big) + m\Big(\bigcup_{j\le N} C_j \setminus A\Big)$$

$$\le m\Big(\bigcup_{j=1}^{\infty} C_j \setminus \bigcup_{j\le N} C_j\Big) + m\Big(\bigcup_{j=1}^{\infty} C_j \setminus A\Big)$$

$$= m\Big(\bigcup_{j>N} C_j\Big) + m\Big(\bigcup_{j=1}^{\infty} C_j \setminus A\Big)$$

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$$\le \sum_{j>N} m(C_j) + \underbrace{m\Big(\bigcup_{j=1}^{\infty} C_j\Big) - m(A)}_{\varepsilon}$$

$$\le \varepsilon + \varepsilon$$

$$= 2\varepsilon. \quad \blacksquare$$

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