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Let $\mathcal{M}$ be collection of all measurable sets in $\mathbb{R}^n$.

- $\emptyset \in \mathcal{M}$.
- $A, B \in \mathcal{M} \Rightarrow A \cup B \in \mathcal{M}$
- $A, B \in \mathcal{M} \Rightarrow B \setminus A \in \mathcal{M}$

This looks like an algebraic structure on $\mathcal{M}$.

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In school algebra, learned how to solve equations.

Abstract algebra is the study of the minimal amount of structure a set needs to have in order to be able to solve equations.

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For example, any set G w/a binary operation $\oplus$ that satisfies

① $x \oplus y \in G$ $\forall x, y \in G$ ("G is closed under $\oplus$")

② $x \oplus y \oplus z$ is unambiguous ("$\oplus$ is associative")

③ $\exists 0 \in G$ s.t. $x \oplus 0 = 0 \oplus x = x$ $\forall x \in G$ ("G has an additive identity")

④ $\forall x \in G$ $\exists y \in G$ s.t. $x \oplus y = 0$ ("Every $x \in G$ has an additive inverse")

is called a "group".

$\div$ not assoc., since $4 \div 2 \div 2$ is ambiguous

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If also want to be able to multiply, can work in a more general set: a ring, w/ is a G w/ $\oplus$ and $\otimes$ s.t. $(G, \oplus)$ satisfy 1–4, and

⑤ $\oplus$ is commutative, i.e. $a \oplus b = b \oplus a$ $\forall a, b \in G$

⑥ G is closed under $\otimes$

⑧ $\exists 1 \in G$ s.t. $1 \neq 0$ and $1 \otimes x = x \otimes 1 = x$

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⑨ Distributive Prop:

$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

and

$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$

⑦ $\otimes$ is associative

If want to divide, study Fields: any G w/ $\oplus$ and $\otimes$ satisfying ①–⑨ as well as

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⑩ $\otimes$ is commutative

⑪ $\forall x \in G \setminus \{0\}$ $\exists y \in G$ s.t. $x \otimes y = 1$.

Back to $\mathcal{M}$:

- $A, B \in \mathcal{M}$, $A \cap B \in \mathcal{M}$.

Is $\mathcal{M}$ a group? a ring? a field?

Not a group under $\cup$: $\emptyset$ is add. id, but $A \cup ? = \emptyset$

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<u>However</u>, under Δ,

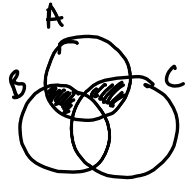
$\mathcal{M}$ <u>is</u> a group!

Moreover, under Δ as $\oplus$ and $\cap$ as $\otimes$, it's ~~almost~~ a ring!

<u>Distribution</u>:

$$A \cap (B \Delta C) \overset{?}{=} (A \cap B) \Delta (A \cap C)$$

2



So distrib. holds!

<u>Mult identity?</u> = $\mathbb{R}^n$

b/c $\mathbb{R}^n \cap A = A$

<u>Note</u>: $\mathcal{M}$ <u>isn't</u> a field, since ~~a type~~

$$A \cap ? = \mathbb{R}^n$$

3

In measure theory, we care especially about open sets, b/c if ~~we~~ know measures of all open sets, then that <u>determines</u> measure of any ~~msble~~ set.

Thus, most interested in spaces of sets which had <u>nice</u> algebraic properties

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and contain all open sets.

More generally, a "<u>σ-algebra</u>" is any collection of sets s.t.

- $\emptyset \in$
- closed under ctble unions
- closed under complement

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The <u>Borel</u> <u>σ-algebra</u> is the smallest σ-algebra containing all open sets.

σ stands for Summe (Union)

Simplest sets in Borel σ-algebra are opens and closeds.

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Recall unions of opens are open and intersections of closeds are closed

Next most complicated type of set after open/closed are

G_δ = ctble intersection of opens

F_σ = ctble union of closeds.

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G ~~[illegible]~~ = Gebiet
"neighborhood"

δ = Durschnitt
"average"
or "intersection"

F = Fermé "closed"

σ = Summe "union"

2

Thm: $A \subseteq \mathbb{R}^n$ is measurable iff
$A = G_\delta \setminus$ set of measure 0
iff $A = F_\sigma \cup$ set of msre 0

To summarize: any measurable set can be approximated by a "nice" set if willing to incur some cost!

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Nice =	Cost =
$G_\delta \supseteq A$	meas = 0
$F_\sigma \subseteq A$	meas = 0
open $\supseteq A$	meas $\leq \varepsilon$
closed $\subseteq A$	meas $\leq \varepsilon$
compact $\subseteq A$	meas $\leq \varepsilon$
Finite union of boxes (both in and out of A)	meas $\leq \varepsilon$

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~~Eg~~
~~Pf~~

Fact: ∃ Lebesgue msble sets that aren't in Borel σ-algebra.

~~But they differ from~~

But any msbl set differs from a set in Borel σ-algebra by measure 0 set.

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