1

On midterm, you proved a version of

<u>Banach-Tarski Thm, 1-d:</u>

(1924)

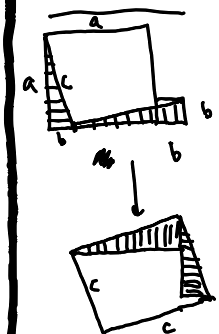
Given intervals $I, J \subseteq \mathbb{R}$. Then $\exists A_k \subseteq \mathbb{R}$ and $\exists \alpha_k \in \mathbb{R}$ s.t.

$$I = \bigsqcup_{k=1}^{\infty} A_k \text{ and } J = \bigsqcup_{k=1}^{\infty} (A_k + \alpha_k)$$

Informally: can break I into

2

ctbly many pieces that, when moved around and reassembled, form J.



$a^2 + b^2 = c^2$

3

S'pose we have a nice measure, i.e.

$$\mu : P(\mathbb{R}) \longrightarrow [0, \infty]$$

s.t. • μ is ctbly additive,

i.e. $\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$

and

• μ is translation invariant,

i.e. $\mu(A + \lambda) = \mu(A)$ $\forall \lambda \in \mathbb{R}$.

4

Pick any intervals $I, J \subseteq \mathbb{R}$

Banach-Tarski $\Rightarrow \exists A_k \subseteq \mathbb{R}$, $\exists \alpha_k \in \mathbb{R}$

s.t. $I = \bigsqcup_{k=1}^{\infty} A_k$, $J = \bigsqcup_{k=1}^{\infty} (A_k + \alpha_k)$

$$\mu(I) = \sum_{k=1}^{\infty} \mu(A_k) = \sum_{k=1}^{\infty} \mu(A_k + \alpha_k) = \mu(J)$$

In fact, $\mu(I) = 0$ or ∞

5

b/c $I = [0,1)$

$J = [1,2]$

$\Rightarrow I \sqcup J = \text{interval} = [0,2]$

$\Rightarrow \mu([0,2]) = \mu([0,1)) + \mu([1,2])$

$= \mu([0,2]) + \mu([0,2])$

$= 2\mu([0,2])$

$\Rightarrow \mu([0,2]) = 0$ or ∞.

$\Rightarrow \mu(I) = 0$ $\forall I$

or

$\mu(I) = \infty$ $\forall I$.

6

We've proved:

<u>Thm</u>: $\nexists$ any non-trivial measure on $\mathbb{R}$ that's simultaneously ctbly add. <u>and</u> translation-invariant.

Outer measure is really close to being perfect! (i.e. having these two props) And when restricted to large class of msble sets, is perfect!

1

Thanks also

Banach-Tarski Thm, 3-d:

If $X, Y \subseteq \mathbb{R}^3$ have non-ϕ interior, then $\exists$ finitely many $A_1, A_2, \dots, A_N \subseteq \mathbb{R}^3$ and $\rho_1, \dots, \rho_N$ rotations and $\alpha_1, \dots, \alpha_N$ translations s.t. $X = \bigsqcup_{k=1}^{N} A_k$ $\quad Y = \bigsqcup_{k=1}^{N} (\rho_k A_k + \alpha_k)$

2

Thus, $\nexists$ non-trivial measure on $\mathbb{R}^3$ that's finitely additive and rotation/translation invariant.

Heuristic "construction" of a non-msble set $\subseteq [0,1]$.

$\forall x \in [0,1]$, flip a coin.
If heads, $x \in E$
If tails, $x \notin E$

3

We expect

$m_*(E) = \frac{1}{2}$ ~~[illegible]~~

Pick any $(a,b) \subseteq [0,1]$.

$m_*(E \cap (a,b)) = \frac{b-a}{2}$

Pick any open $\mathcal{O} \subseteq [0,1]$.

$m_*(E \cap \mathcal{O}) = \frac{1}{2} m_*(\mathcal{O})$

b/c $\mathcal{O} = \bigsqcup_{k=1}^{\infty} (a_k, b_k)$ $\forall$ open $\mathcal{O}$

I claim can't approx. E by any open set.

4

$m_*(\mathcal{O} \setminus E) = m_*(\mathcal{O} \cap E^c)$

If E is msble, Caratheodory

$= m_*(\mathcal{O}) - m_*(\mathcal{O} \cap E)$

$= m_*(\mathcal{O}) - \frac{1}{2} m_*(\mathcal{O})$

$= \frac{1}{2} m_*(\mathcal{O})$

If $\mathcal{O} \supseteq E$, then

$\geq \frac{1}{2} m_*(E) = \frac{1}{4}$.

So can't approximate E by open set.

5

An actual construction of a non-msble set

Recall you obtained (on midterm) a set $X \subseteq [0,1]$ s.t. $(X - X) \cap \mathbb{Q} = \{0\}$.

and $\bigsqcup_{q \in \mathbb{Q}} (X + q) = \mathbb{R}$

Thus, $\infty = m_*(\mathbb{R}) = m_*\left(\bigsqcup_{q \in \mathbb{Q}} (X+q)\right)$

$\leq \sum_{q \in \mathbb{Q}} m_*(X + q)$

6

$\Rightarrow \exists q \in \mathbb{Q}$ s.t.

$m_*(X) = m_*(X + q) \neq 0$

$\Rightarrow m_*(X) > 0$.

By problem 29, if X is msble and $m(X) > 0$, then $X - X \supseteq (-\delta, \delta)$, $\delta > 0$.

But $\mathbb{Q}$ is dense in $\mathbb{R}$, so $\exists q \in (0, \delta) \cap \mathbb{Q}$

$\Rightarrow q \in \mathbb{Q} \cap (X - X) = \{0\}$. $\Rightarrow\!\Leftarrow$

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This X, which each of could choose differently, is an example of a

Vitali set.

<u>Construction of Vitali set</u>

Say $x \sim y$ iff $x - y \in \mathbb{Q}$.

Let $[x] := \{y \in [0,1) : y \sim x\}$.

<u>Lemma</u>: $[x] \neq [y] \Rightarrow [x] \cap [y] = \emptyset$.

2

Note that

$$I := [0,1) = \bigcup_{\alpha \in I} [\alpha]$$

Some of these sets $[\alpha]$ are redundant — remove them!

Leaves

$$I = \bigsqcup_{\alpha \in X} [\alpha]$$

In other words, we choose one element from each distinct equivalence class; call collection

3

of these elements X.

This uses the Axiom of Choice, which informally states that given any collection of non-empty sets, you can choose one element from each set.

Let's state more formally:

4

<u>Axiom of Choice</u>:

Given a set U ("universe")

Given a collection ~~[illegible]~~

$$\Sigma \subseteq \mathcal{P}(U) \text{ s.t.}$$

$\emptyset \notin \Sigma$. Then

$\exists$ function $f : \Sigma \to U$

s.t. $f(\sigma) \in \sigma$. $\forall \sigma \in \Sigma$.

5

To construct our Vitali set, let $\Sigma := \{[\alpha] : \alpha \in I\}$

Let $X := f(\Sigma)$

where f ~~[illegible]~~ is a choice function.

Any such Vitali set has positive exterior measure <u>and</u> is non-msble.

6