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TA applications are due @ noon on Tuesday (info on math/stat website)

Integration theory

Recall: measuring the size of a set is related to integration: given $A \subseteq \mathbb{R}$, let χ_A

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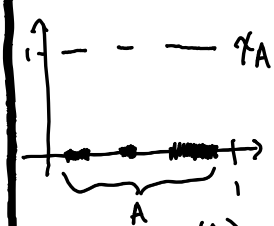
be its characteristic function, i.e.

$$\chi_A(t) := \begin{cases} 1 & t \in A \\ 0 & t \notin A \end{cases}$$



Expect: $m(A) = \int_0^1 \chi_A$

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This works, for example, when A is an interval. But this expectation is false even for relatively tame sets, e.g. $A = \mathbb{Q} \cap [0,1]$.

Let's see why:

The Riemann integral

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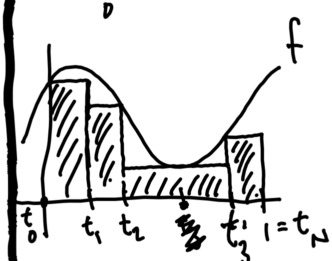
Idea: how Riemann would calculate

$$\int_0^1 f:$$



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Step 1: Pick a partition of $[0,1]$:

$$P := \{t_0 = 0 < t_1 < t_2 < \dots < t_N = 1\}$$

Step 2: Construct the "Lower Riemann sum":

$$L(f,P) := \sum_{k=1}^{N} m_k \cdot (t_k - t_{k-1})$$

where $\cancel{m_k = \inf\{f(x) : x \in [t_k, t_k}$

$$m_k := \inf\{f(x) : x \in [t_{k-1}, t_k]\}.$$

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Step 3: Construct "Upper Riemann Sum"

$$U(f,P) := \sum_{k=1}^{N} M_k (t_k - t_{k-1})$$

where

$$M_k := \sup\{f(x) : x \in [t_{k-1}, t_k]\}$$

~~Step 4: To evaluate~~ $\cancel{\int_0^1 f}$, ~~find~~ $\cancel{P}$ ~~s.t.~~ $\cancel{L(f,P) = U(f,P) =: \int_0^1 f}$

Might not happen ☹

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Step 4:

Lemma: $\forall$ partitions P, Q,

$$L(f,P) \leq U(f,Q)$$

Thus, $\sup\{L(f,P) : P = \text{part.}\}$
$\leq \inf\{U(f,P) : P = \text{part}\}$

If equality holds, we say f is Riemann integrable and define $\int_0^1 f := \sup\{L(f,P)\}$.

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Example: $\chi_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable:

b/c $\mathbb{Q}$ is dense in $\mathbb{R}$, and $\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$,

$$L(\chi_{\mathbb{Q}\cap[0,1]}, P) = 0 \quad \forall P$$

$$U(\chi_{\mathbb{Q}\cap[0,1]}, P) = 1 \quad \forall P$$

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Ben's Q:

Consider $B_N := \{\frac{a}{n} \in [0,1] : a \in \mathbb{Z},\ 1 \leq n \leq N\}$

This is finite, hence χ_{B_N} is Riemann integrable

and $\int_0^1 \chi_{B_N} = 0 \quad \forall N$

But $\lim_{N\to\infty} \chi_{B_N}(t) = \chi_{\mathbb{Q}\cap[0,1]}(t)$
$\forall t \in [0,1]$

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Thus,

$$\lim_{N\to\infty} \int_0^1 \chi_{B_N} \neq \int_0^1 \lim_{N\to\infty} \chi_{B_N}$$

b/c this d.n.e.

This is an example of an evil LEO: Riemann integrals don't play nice w/ limits.

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Who cares!?

Anybody who works w/ Fourier analysis.

$\boxed{\text{A crash course in Fourier series}}$

~~Four~~ Object of Fourier analysis is to decompose a given ~~

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periodic f^n into its constituent pure waves. We'll express this in terms of complex exponentials.

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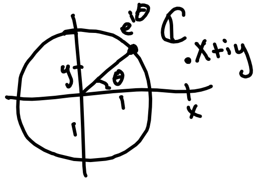
Recall $e^{i\theta} := \cos\theta + i\sin\theta$.



Very nice way to simplify notation:

$$e(\alpha) := e^{2\pi i\alpha}$$

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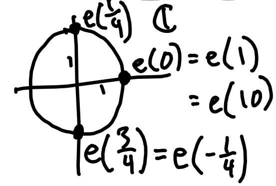

Note that $e(x)$ is 1-periodic.

$$\sin(2\pi x) = \frac{e(x)-e(-x)}{2i}$$

$$5 = e(0)$$

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Our goal: given a 1-periodic fn. f, to find $a_n \in \mathbb{C}$ s.t.

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e(nx)$$

Can this always be done?

Key insight: notice that

$$\int_0^1 e(kx)\,dx = \frac{e(kx)}{2\pi i k}\Big|_0^1 = 0$$

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So $\forall k \in \mathbb{Z}$,

$$\int_0^1 e(kx)\,dx = \begin{cases} 0 & \text{if } k\neq 0 \\ 1 & \text{if } k=0 \end{cases}$$

Thus, if we <u>could</u> write $f(x) = \sum_{n\in\mathbb{Z}} a_n e(nx)$, then

$$\int_0^1 f(x)e(-kx)\,dx =$$

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$$= \int_0^1 \sum_{n\in\mathbb{Z}} a_n e(nx)\, e(-kx)\,dx$$

$$= \sum_{n\in\mathbb{Z}} a_n \int_0^1 e(nx)e(-kx)\,dx$$

$$= \sum_{n\in\mathbb{Z}} a_n \int_0^1 e\big((n-k)x\big)\,dx$$

$$= a_k \cdot 1 = a_k$$

Thus, we expect:

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<u>Conj</u>: If f is 1-periodic, then

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e(nx)$$

where $a_n := \int_0^1 f(x)e(-nx)\,dx$

This turns out to be wrong in general.

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The main ~~step~~ dubious step in our argument:

$$\int_0^1 \sum_{n\in\mathbb{Z}} a_n e(nx)\, e(-kx)\,dx$$

$$= \int_0^1 \lim_{N\to\infty} \sum_{|n|\le N} a_n e(nx) \cdot \sim$$

Questionable

$$= \lim_{N\to\infty} \int_0^1 \sum_{|n|\le N} a_n e(nx) \sim$$

the rest is OK...

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This is why we would like Riemann integral to play nice w/limits!

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