**1**

Last time we saw limitations of Riemann integration, e.g.

$$\int_0^1 \chi_{\mathbb{Q}} \text{ doesn't exist}$$

even though intuitively it should exist and $=0$.

To be fair, $\chi_{\mathbb{Q}}$ is discontinuous everywhere.

**2**

**Prop$^{\underline{n}}$**: If $f:[0,1]\to\mathbb{R}$ is cts on $[0,1]$, then

$$\int_0^1 f \text{ exists.}$$

Riemann.

**Pf idea**: If $f$ is cts on $[0,1]$, then it's uniformly cts on $[0,1]$. Thus ~~cts~~ can partition $[0,1]$ into finitely many pieces

**3**

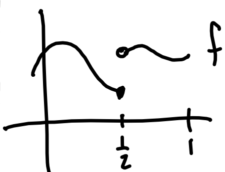
s.t. $f$ is almost constant on each piece.

$\Rightarrow U(f,P) - L(f,P) < \varepsilon$.

$\Rightarrow \int_0^1 f$ exists. "QED"

But there are Riemann integrable $f^{\underline{n}}$s that aren't continuous, e.g.

**4**




is Riemann integrable.

If $f$ has finitely many discontinuities, then $f$ is Riemann integrable.

~~If spread out~~

**Conj**: If discontinuities are "spread out" enough?

**5**

**Thm (Lebesgue's Criterion)**:

$f:[0,1]\to\mathbb{R}$ is Riemann integrable iff the set of all ~~discontin~~ discontinuities of $f$ in $[0,1]$ has measure $0$.

Thus, e.g.

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{q} & x = \frac{a}{q} \text{ red} \end{cases}$$

**6**

is continuous @ all irrationals, discontinuous @ all rat'ls, and therefore Riemann integrable on $[0,1]$.

Proof of Lebesgue's Criterion on next problem set.

**Language**: If $f$ has property $P$ except @ set of measure $0$, $f$ has $P$ almost everywhere.

**1**

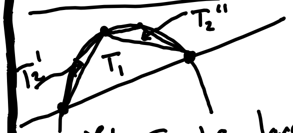
e.g. Lebesgue says $f$ is Riemann integrable iff $f$ is continuous a.e.

This means that many $f^{n}$s that aren't Riemann integrable.

**2**

How do we approach integration using a method other than Riemann's?

Archimedes: (Αρχιμήδης)




Let $T_1$ be largest inscribable triangle.

**3**

Repeat in smaller sections to get $T_2'$, $T_2''$.

Archimedes proved that

$$\frac{1}{4} \text{Area}(T_1) = \text{Area}(T_2') + \text{Area}(T_2'')$$

Thus,
Area of Parabola =

$$\text{Area}(T_1) + \frac{1}{4}\text{Area}(T_1) + \frac{1}{4^2}\text{Area}(T_1) + \cdots$$

**4**

$$= \frac{4}{3} \text{Area}(T_1)$$

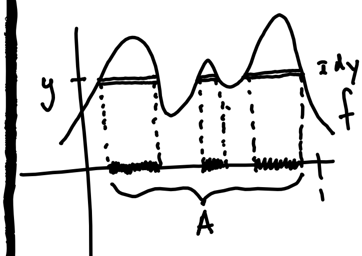
Different integrations:

Archimedes: Fill area w/ triangles.

Riemann: Fill area w/ vertical slices.

Lebesgue: Fill area w/ horizontal slices.

**5**




Area of slice = $m(A)\, dy$

What is $A$? (in terms of $f$, $y$)

$$A = \{x : f(x) \geq y\}$$

**6**

Thus we might guess that

$$\int_0^1 f = \int_0^\infty m(\{x \in [0,1] : f(x) \geq y\})\, dy$$

a Riemann integral

B/C laziness, I'll write

$$\{f \geq y\} := \{x : f(x) \geq y\}.$$

This approach only works when $f \geq 0$.

1

Example:

$$\int_0^1 \chi_{\mathbb{Q}} = \int_0^\infty m(\chi_{\mathbb{Q}} \ge y)\, dy$$

$m(\{\chi_{\mathbb{Q}} \ge 2\}) = 0$

$m(\{\chi_{\mathbb{Q}} \ge 1\}) = 0$

$m(\{\chi_{\mathbb{Q}} \ge \frac{1}{1000}\}) = 0$

$$\le \int_0^\varepsilon 1 \, dy = \varepsilon$$

2

Thus, $\int_0^1 \chi_{\mathbb{Q}} = 0.$

---

The Riemann integral works for cts f$^{n}$s.

~~When~~ What are the nice f$^{n}$s for which Lebesgue integration should work?

Def$^{n}$: We say $f: E \to \mathbb{R}$ (where $E \subseteq \mathbb{R}^n$ msble) is

3

measurable iff $\{f \ge y\}$ is msble $\forall y \in \mathbb{R}$.

iff $f^{-1}([y,\infty))$ is msble $\forall y \in \mathbb{R}$

4

Prop$^{n}$: $f: E \to \mathbb{R}$ is msble iff

① $\{f \ge y\} \in \mathcal{M}$ $\forall y \in \mathbb{R}$

② $\{f > y\} \in \mathcal{M}$ $\forall y \in \mathbb{R}$

③ $\{f \le y\} \in \mathcal{M}$ $\forall y \in \mathbb{R}$

④ $\{f < y\} \in \mathcal{M}$ $\forall y \in \mathbb{R}$

⑤ $\{a < f < b\} \in \mathcal{M}$ ~~$\forall$ non-empty~~ $\forall (a,b) \subseteq \mathbb{R}$.

5

6