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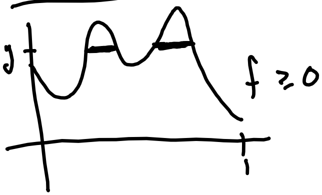
Last time:



$$\int_0^1 f = \int_0^\infty m(\{f \geq y\})\,dy$$

Lebesgue

Riemann

~~where $\{f \geq y\} := \{x \in [0,1$~~

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where

$$\{f \geq y\} := \{x \in [0,1] : f(x) \geq y\}$$

$$= f^{-1}([y,\infty))$$

(assuming $f : [0,1] \to \mathbb{R}_{\geq 0}$)

For Lebesgue's approach to work well, we want

$\{f \geq y\} \in \mathcal{M}$ [= set of all Lebesgue measurable sets) $\forall y$

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Def$^{\underline{n}}$: A ~~measurable~~ (Lebesgue) measurable function is $f : E \to \mathbb{R}$ s.t. $E \in \mathcal{M}$ and

$$f^{-1}([y,\infty)) \in \mathcal{M} \quad \forall y \in \mathbb{R}.$$

Prop$^{\underline{n}}$: f msble iff

$\{f \geq y\} \in \mathcal{M} \quad \forall y \in \mathbb{R}$

$\iff \{f > y\} \in \mathcal{M} \quad \forall y$

$\iff \{f \leq y\} \in \mathcal{M} \quad \forall y$

$\iff \{f < y\} \in \mathcal{M} \quad \forall y$

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$\iff \{a < f < b\} \in \mathcal{M} \quad \forall a,b \in \mathbb{R}.$

Pf:

① Suppose $\{f \geq y\} \in \mathcal{M} \ \forall y$.

$\Rightarrow \{f > y\} \in \mathcal{M} \ \forall y$

b/c ~~$\{f \geq y\} \in \bigcap \{f \geq y - \frac{1}{n}\}$~~

$$\{f > y\} = \bigcup_{n=1}^{\infty} \{f \geq y + \tfrac{1}{n}\} \in \mathcal{M}$$

...

∎

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Riemann $\int$: cts. f$^{\underline{n}}$s

as Lebesgue $\int$: msble f$^{\underline{n}}$s

To explore this analogy, we first recall some true facts about continuous f$^{\underline{n}}$s.

Given cts. $f : X \to Y$.

Q: Does f preserve openness? i.e.

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if $\mathcal{O} \subseteq X$ is open, must $f(\mathcal{O})$ be open?

No: $f : \mathbb{R} \to \mathbb{R}$

$f(x) = 1 \ \forall x$

$\Rightarrow f((0,1)) = [1,1].$

Q: Must f preserve closedness?

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<u>No</u>: $f(x) = \frac{1}{x}$ on $[1, \infty)$

- $f(x) = e^x$

  $f((-\infty, 0]) = (0, 1]$.

- $\arctan(\mathbb{R}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

<u>Fact</u>: cts f^ns preserve compactness

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Looks like cts f^ns don't play well w/ topological properties,

<u>BUT</u>

<u>Prop^n</u>: $f : X \longrightarrow Y$ is cts. $\iff f^{-1}(\mathcal{O}) = \text{open}$ $\forall$ open $\mathcal{O}$.

$\iff f^{-1}(C) = \text{closed}$ $\forall$ closed C.

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<u>Q</u>: $f^{-1}(K) = \text{compact}$ $\forall$ compact K?

<u>No</u>: $f(x) = 1$ $\forall x \in \mathbb{R}$

$\Rightarrow f^{-1}([1,1]) = \mathbb{R}$.

$$f : (0,1) \longrightarrow \mathbb{R}$$
$$x \longmapsto 1$$

$$f^{-1}([1,1]) = (0,1)$$

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<u>Pf of Prop^n</u>:

Pick $\mathcal{O}$ open.

Pick $x \in f^{-1}(\mathcal{O})$.

<u>Want</u>: $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(\mathcal{O})$

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Since $\mathcal{O}$ is open,

$\exists \varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \subseteq \mathcal{O}$

$\Rightarrow \exists \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ by cty of f.

$\Rightarrow f(B_\delta(x)) \subseteq \mathcal{O}$

$\Rightarrow B_\delta(x) \subseteq f^{-1}(\mathcal{O})$. //

This proves $\Rightarrow$.

$\Leftarrow$ is exercise. ∎

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<u>Prop^n</u>: f is msble

<u>iff</u> $f^{-1}(\mathcal{O}) \in \mathcal{M}$ $\forall$ open $\mathcal{O}$

<u>iff</u> $f^{-1}(C) \in \mathcal{M}$ $\forall$ closed C.

<u>Corollary</u>: f cts $\Rightarrow f$ msble.

Other analogues of cts properties:

<u>Prop^n</u>: Finite sums and products of msble f^ns are msble.

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Quasi-analogues of cts props

Recall if f, g cts, then $f \circ g$ is cts.

What about if f, g msble, must $f \circ g$ msble?

Half-yes:
It's possible for msble $\circ$ cts $\neq$ msble.

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But

Prop^{n}: cts $\circ$ msble $=$ msble.

More precisely:

If $f : \mathbb{R} \to \mathbb{R}$ cts and $g : \mathbb{R}^n \to \mathbb{R}$ msble then $f \circ g : \mathbb{R}^n \to \mathbb{R}$ is msble.

Pf: E.T.S. $(f \circ g)^{-1}(\mathcal{O}) \in \mathcal{M}$ $\forall$ open $\mathcal{O}$.

i.e. $g^{-1}(f^{-1}(\mathcal{O})) \in \mathcal{M}$

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And it is!

b/c f cts
$\Rightarrow f^{-1}(\mathcal{O}) =$ open
$\Rightarrow g^{-1}(f^{-1}(\mathcal{O})) \in \mathcal{M}$. ∎

Non-analogues of cts f^{n} props

Q: Given $f_n : [0,1] \to \mathbb{R}$ cts s.t. $f(x) := \lim_{n\to\infty} f_n(x)$ $\forall x$ exists $\forall x \in [0,1]$. Must

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f be cts on $[0,1]$?

No: $f_n(x) := x^n$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 1 & x=1 \\ 0 & — \end{cases}$$

$\forall x \in [0,1]$ $\quad = \chi_{\{1\}}$

which isn't cts on $[0,1]$.

What if we weaken the hypothesis?

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Q: S'pose $f_n : [0,1] \to \mathbb{R}$ cts

s.t. $\forall x \in [0,1]$,
$f(x) := \sup\{f_n(x)\}$

Must f be cts on $[0,1]$?

No: $f_n(x) := 1 - x^n$,

then $\sup f_n(x) = \chi_{[0,1)}(x)$

which isn't cts.

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Also, cty isn't preserved under lim sup or lim inf.

But:

Prop^{n}: If $\{f_n\}$ msble

then
$\sup f_n(x)$, $\inf f_n(x)$,
$\limsup_{n\to\infty} f_n(x)$, $\liminf_{n\to\infty} f_n(x)$,
$\lim_{n\to\infty} f_n(x)$ are all msble when they exist.