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Last time:

Analogies between cts. f^ns and msble f^ns

Prop^n: f msble iff $f^{-1}(\mathcal{O}) \in \mathcal{M}$ $\forall$ open $\mathcal{O}$ iff $f^{-1}(C) \in \mathcal{M}$ $\forall$ closed C

Cor: f cts $\Rightarrow$ f msble.

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Converse is false:

$\chi_{[0,1]}$ is msble, but not cts.

Pf:

$$\{\chi_{[0,1]} \geq y\} = \begin{cases} \emptyset & y > 1 \\ [0,1] & 0 < y \leq 1 \\ \mathbb{R} & y \leq 0 \end{cases}$$

Since $\emptyset, [0,1], \mathbb{R} \in \mathcal{M}$, $\chi_{[0,1]}$ is msble. $\blacksquare$

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More generally, when is χ_A msble?

$$\{\chi_A \geq y\} = \begin{cases} \emptyset & y > 1 \\ A & 0 < y \leq 1 \\ \mathbb{R} & y \leq 0 \end{cases}$$

So:

Prop^n: χ_A is msble f^n **iff** A is a msble set.

(Hence name "msble f^n".)

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Quasi-analogous:

Prop^n: cts $\circ$ msble $=$ msble.

But opposite is false!

Non-analogous:

Prop^n: Given any sequence of msble f^ns $\{f_n\}$, then $\sup_n \{f_n\}$ is msble.

$\inf_n \{f_n\}$ is msble.

$\lim_{n\to\infty} \sup \{f_n\}$ is msble.

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$\liminf_{n\to\infty} f_n$ is msble.

$\lim_{n\to\infty} f_n$ is msble (assuming exists)

Q: Do msble f^ns map msble sets to msble sets?

No! Even a cts f^n doesn't do this!

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e.g. if N is Vitali set and $F: \mathbb{R} \to [0,1]$ is Cantor-Lebesgue,

then $m(F^{-1}(N)) = 0$

$\Rightarrow F^{-1}(N) \in \mathcal{M}$

but $F(F^{-1}(N)) = N \notin \mathcal{M}$.

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Q: If f msble, must $f^{-1}(\text{msble set}) = \text{msble set}$?

~~$f(x) = 0 \ \forall x$~~

$f: \mathbb{N} \to \mathbb{R}$

$x \mapsto 0$

?

Then $f^{-1}(0) = \mathbb{N}$

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What about $f: \mathbb{R} \to \mathbb{R}$?

No: $\exists$ msble $f: \mathbb{R} \to \mathbb{R}$ and $A \in \mathcal{M}$ s.t.

$f^{-1}(A) \notin \mathcal{M}$.

This is annoying b/c in the literature, $f: X \to Y$ is defined to be msble iff

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$f^{-1}(\text{msble set})$ is msble set.

$\boxed{\text{WTF?}}$

In general, given $f: X \to Y$ to discuss measurability we need to know which sets in X and Y we're allowed to measure.

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In general, a measure space is any set X w/ a set $\Sigma \subseteq P(X)$ s.t. Σ = σ-algebra.

So when discussing msble fns, we need

$f: (X, \Sigma) \to (Y, T)$

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Defn: f is msble iff $f^{-1}(\text{msble set}) = \text{msble set}$

i.e. $f^{-1}(A) \in \Sigma$ $\forall A \in T$.

Our defn in the context of $f: \mathbb{R} \to \mathbb{R}$ isn't $f: (\mathbb{R}, \mathcal{M}) \to (\mathbb{R}, \mathcal{M})$.

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It's actually

$f: (\mathbb{R}, \mathcal{M}) \to (\mathbb{R}, \mathcal{B})$

Borel σ-algebra

When we say f is msble, we're really saying f is "Lebesgue-to-Borel" msble.

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This asymmetry causes some problems, e.g. cts $\circ$ msble $=$ msble but msble $\circ$ cts $\neq$ msble; we don't have such nonsense in msble f$^{\underline{n}}$s $(\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$.

But Lebesgue-to-Borel

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f$^{\underline{n}}$s are a larger class of f$^{\underline{n}}$s and Lebesgue integration works for all these.

Back to our down-to-earth def$^{\underline{n}}$ of msble f$^{\underline{n}}$...

Continuity analogies b/w cts. and msble:

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Thm (Cauchy, 1821):

S'pose $f_n : \mathbb{R} \to \mathbb{R}$ all cts, and that $\sum_{n=1}^{\infty} f_n(x)$ exists $\forall x \in \mathbb{R}$.

Then $\sum_{n=1}^{\infty} f_n(x)$ is cts.

Pf idea: Let $S_N(x) := \sum_{n \leq N} f_n(x)$

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$R_N(x) := \sum_{n > N} f_n(x)$.

Want: $S_N(x) + R_N(x)$ is cts. @ a, i.e.

WTF $\delta > 0$ s.t. $|x - a| < \delta$

$$\Rightarrow \underbrace{\left|\left(S_N(x) + R_N(x)\right) - \left(S_N(a) + R_N(a)\right)\right|} < \varepsilon.$$

$$= |S_N(x) - S_N(a) + R_N(x) - R_N(a)|$$

$$\leq |S_N(x) - S_N(a)| + |R_N(x)| + |R_N(a)|$$

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Cauchy's Proof:

Given $\varepsilon > 0$.

Step 1: $S_N(x)$ is cts. @ a

$\Rightarrow \exists \delta > 0$ s.t.

$|x - a| < \delta \Rightarrow |S_N(x) - S_N(a)| < \frac{\varepsilon}{3}$

Step 2: $\forall x$, $\lim_{N \to \infty} S_N(x)$ exists

$\Rightarrow \lim_{N \to \infty} R_N(x) = 0$, i.e.

$\exists M$ s.t. $N > M \Rightarrow |R_N(x)| < \frac{\varepsilon}{3}$

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Step 3: WIN!

Pick any $x \in (a - \delta, a + \delta)$.

Step 1 $\Rightarrow |S_N(x) - S_N(a)| < \frac{\varepsilon}{3}$

Step 2 $\Rightarrow \exists M$ s.t.

$|R_N(x)| < \frac{\varepsilon}{3}$

$\forall N > M$

And $\exists M'$ s.t. $|R_N(a)| < \frac{\varepsilon}{3}$

$\forall N > M'$.

Thus $|S_N(x) + R_N(x) - (S_N(a) + R_N(a))|$

$< \varepsilon$. $\blacksquare$

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Cauchy was wrong!

Abel discovered a counterexample,

<u>BUT</u>: can be fixed.

<u>Q</u>: How can we fix Cauchy's proof?

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