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Last time:

We proved Cauchy's false theorem:

Thm: If $f_n : \mathbb{R} \to \mathbb{R}$ cts and $\sum_{n=1}^{\infty} f_n(x)$ exists $\forall x$, then $\sum_{n=1}^{\infty} f_n(x)$ is cts.

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Proof: Given $\varepsilon > 0$.

Let $S_N(x) := \sum_{n \le N} f_n(x)$

and $R_N(x) := \sum_{n > N} f_n(x)$

Want: $S_N + R_N$ is cts @ a.

Step 1: S_N is cts. @ a

$\Rightarrow \exists \delta > 0$ s.t. $\forall x \in (a-\delta, a+\delta)$

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we have

$|S_N(x) - S_N(a)| < \varepsilon/3$.

Step 2: $R_N(x) \xrightarrow{N \to \infty} 0$

So $\exists M$ s.t. $|R_N(x)| < \frac{\varepsilon}{3}$ $\forall N > M$.

Step 3: WIN:

$|S_N(x) + R_N(x) - (S_N(a) + R_N(a))|$

$=$

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$\le |S_N(x) - S_N(a)| + |R_N(x)| + |R_N(a)|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$

$\forall x \in (a-\delta, a+\delta)$ and $\forall N > M$

$\Rightarrow S_N + R_N$ cts. @ a.

"∎"

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What's wrong w/ the proof?

Issue: δ and M secretly depend on other parameters:

$\delta = \delta(a, \varepsilon, N)$

$M = M(\varepsilon, x, a)$

Start w/ ε, a.

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What do we choose next?

Option 1: Choose δ.

Can't choose δ w/out knowing N, and can't know N w/out knowing M. ☹

Option 2: Choose M first.

M depends on x, which is from $(a-\delta, a+\delta)$, so need to know δ! ☹

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To fix the issue, we need ~~to~~ an M that's independent of x:

then, given a, ε, we choose $M = M(\varepsilon, a)$

then choose $N > M$,

then choose $\delta = \delta(\varepsilon, a, N)$.

In other words, to make Cauchy's proof work, we need to

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know:

$$\boxed{\forall \varepsilon > 0,\ \exists M \text{ s.t. } |R_N(x)| < \varepsilon \quad \forall N > M \text{ and } \forall x.}$$

By contrast, $R_N(x) \xrightarrow{N\to\infty} 0$ means

$$\boxed{\forall \varepsilon > 0,\ \forall x,\ \exists M \text{ s.t. } |R_N(x)| < \varepsilon \quad \forall N > M.}$$

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This is the def^{n} of uniform convergence! More precisely:

<u>Def^{n}</u>: Given sequence of f^{n}s $f_n : E \to \mathbb{R}$.

- $f_n \to f$ <u>"pointwise"</u>

<u>iff</u> $\forall x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$

<u>iff</u> $\forall x \in E$, $\forall \varepsilon > 0$ $\exists N$ s.t. $n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$.

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- $f_n \to f$ <u>"uniformly"</u>

iff $\forall \varepsilon > 0$ $\exists N$ s.t. $n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$ $\forall x \in E$.

Uniform Convergence is much stronger than ptwise convergence: roughly speaking, if $f_n \to f$ uniformly

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then $f_n(x) \to f(x)$ @ the same rate everywhere in E.

Example of $f_n \to f$ ptwise but <u>not</u> uniformly

$f_n : [0,1] \to \mathbb{R}$

$x \mapsto x^n$

<u>Recall</u>: $f_n \to \chi_{\{1\}}$ ptwise

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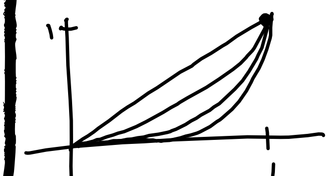

Why not uniform?

~~As appro~~ As $x \to 1^-$, derivative of $f_n(x)$ gets larger and larger, so approaches $\chi_{\{1\}}$ faster and faster.

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More precisely:

I claim that $\exists$ arbitrarily large n for which
$$\left|\chi_{\{1\}}(x) - f_n(x)\right| > \frac{1}{10}$$
somewhere.

In p'ticular, pick any n; then $\exists\, x_0 = \frac{1}{2^{1/n}}$ s.t. $f_n(x_0) = \frac{1}{2}$.

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Cauchy's proof yields:

<u>Thm</u>: If $f_n : E \to \mathbb{R}$ are all cts on E and $f_n \to f$ uniformly on E, then f is cts. on E.

You're used to proving exact results, e.g.

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<u>Thm</u>: If $m(E) > 0$, then $E - E$ contains an open interval centered @ 0.

Analysis allows us to prove results that are precise but not exact, e.g.

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<u>Thm</u>: If $m_*(E) > 0$, then E contains $\geq 99.9\%$ of an open interval, i.e. $\exists I$ s.t.
$$\frac{m_*(E \cap I)}{m_*(I)} \geq 0.999$$

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Three famous examples of this are "<u>Littlewood's Principles</u>"

1) Any set $\left(\begin{smallmatrix}\text{of finite}\\ \text{measure}\end{smallmatrix}\right)$ is pretty much a finite union of cubes.

2) Any (msble) $f^{\underline{n}}$ is pretty much continuous.

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3) If $f_n \to f$ ptwise (and f_n = msble) then the convergence is pretty much uniform.

$\chi_{\mathbb{Q}}$ is msble but discontinuous <u>everywhere</u>!

But $\chi_{\mathbb{Q}}\big|_{\mathbb{R} \setminus \mathbb{Q}}$ is cts.