Littlewood's Principles

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① Any set is pretty much a finite union of cubes

② Any fn is pretty much continuous

③ If $f_n \longrightarrow f$, then convergence is pretty much uniform.

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We've already proved

①: If $m(A) < \infty$, then $\exists$ a finite union of cubes, say, F, s.t. $m(A \triangle F) < \varepsilon$.

The other two:

② Lusin's Thm (1912)

If $f : E \longrightarrow \mathbb{R}$ msble and $m(E) < \infty$, then $\exists$ closed $F \subseteq E$ s.t.

3

$m(E \setminus F) < \varepsilon$ and

$f|_F$ is cts.

CAUTION: $\chi_{\mathbb{Q} \cap [0,1]}$ is discts. @ every pt

BUT $\chi_{\mathbb{Q} \cap [0,1]}\big|_{[0,1] \setminus \mathbb{Q}}$ is cts everywhere.

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Formal version of ③:

Egorov's Thm

(proved by Severini in 1910)

Given msble $f_n : E \longrightarrow \mathbb{R}$ w/ $m(E) < \infty$. If $f_n \longrightarrow f$ a.e. then $\exists$ closed $F \subseteq E$ s.t. $m(E \setminus F) < \varepsilon$ and $f_n \longrightarrow f$ uniformly on F.

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Note: all three thms require domain has finite measure.

Is this hypothesis necessary?

① Yes: necessary! ~~All of~~ All of $\mathbb{R}$ can't be almost covered by finitely many cubes.

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② Not necessary?

③ If yes, need unctbly many discontinuities ...

For ③, $m(E)$ must be finite b/c of the traveling saleswave:

$f_n := \chi_{[n, n+1]}$

$f_n \longrightarrow 0$ ptwise.

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Егоров: advised PhD of Лузин

(Lusin has 6500 descendants)

Some of his students:

- Alexandrov (topology)

- Bari (Fourier series)
- Khinchin (probability, # theory)
- Kolmogorov (probability, CS)

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- Novikov (combinatorial gp theory)
- Schnirelmann (Number theory)
- Urysohn (analyst)
- Lyapunov (cybernetics, math bio, math linguistics)

Proofs:

Lusin: msble $f: E \to \mathbb{R}$ $\Rightarrow$ f pretty much cts.

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Egorov: if $f_n \to f$ a.e. then pretty much uniform.

Idea for proof of Egorov

Given $f_n : E \to \mathbb{R}$ msble. $m(E) < \infty$, $f_n \to f$ ~~a.e.~~

WTF closed $F \subseteq E$ s.t. $m(E \setminus F) < \varepsilon$ and s.t. $|f_n(x) - f(x)| <$ tiny on all of F once n large enough.

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Let

$A_1 := \{x \in E : |f_n(x) - f(x)| < \varepsilon \ \forall n\}$

$A_2 := \{x \in E : |f_n(x) - f(x)| < \varepsilon \ \forall n \geq 2\}$

$A_3 := \{x : |f_n(x) - f(x)| < \varepsilon \ \forall n \geq 3\}$

$\vdots$

E

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More generally:

$A_N := \{x : |f_n(x) - f(x)| < \varepsilon \ \forall n \geq N\}$.

So $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$

$A_N \nearrow E$ b/c

Pick $x \in E$. We know $f_n(x) \to f(x)$

$\Rightarrow \exists N$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

$\Rightarrow x \in A_N$

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Thus ~~?~~ 3.3/3.4

$m(A_n) \longrightarrow m(E)$

$\Rightarrow$ ~~$\forall \varepsilon > 0$~~ $\forall \delta > 0 \ \exists n$ s.t. $m(E \setminus A_n) < \delta$

Summary: We've found an $A_N \subseteq E$ that's huge and $\forall x \in A_N$ we have

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N.$$

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Problem: ~~E~~ to converge uniformly on A means $\forall x \in A$ and $\forall \varepsilon > 0$,

$$|f_n(x) - f(x)| < \varepsilon$$

for large n.

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