

Egorov's Thm.

Given measurable $f_n: E \rightarrow \mathbb{R}$
where $m(E) < \infty$.

If $f_n \rightarrow f$ a.e.,
then \exists closed $F \subseteq E$
s.t. $m(E \setminus F) < \varepsilon$ and
 $f_n \rightarrow f$ uniform on F .

Proof idea: Pick tiny
 $\delta > 0$.

Want $|f_n(x) - f(x)| < \delta$

\forall suff. large n
and for most $x \in E$.

We can do this!

Let $A_1 := \{x: |f_1(x) - f(x)| < \delta, \forall n \geq 1\}$

$A_2 := \{x: |f_2(x) - f(x)| < \delta, \forall n \geq 2\}$
:

We saw that $A_k \nearrow E$.

MCT $\Rightarrow m(A_k) \rightarrow m(E)$.

Thus $\exists N$ s.t.

$$m(E) - m(A_N) < \varepsilon.$$

$$\Rightarrow m(E \setminus A_N) < \varepsilon$$

$$\text{and } |f_n(x) - f(x)| < \delta \quad \forall n \geq N$$

and $\forall x \in A_N$.

Finally, \exists closed $F \subseteq A_N$

s.t. $m(A_N \setminus F) < \varepsilon$.

Thus, $m(E \setminus F) < 2\varepsilon$

$$\text{and } |f_n(x) - f(x)| < \delta \quad \forall x \in F, \forall n \geq N.$$

This doesn't prove 4

Egorov, b/c our choice
of F depends on δ .

~~Thus, $f_n(x) \rightarrow f(x)$ uniform~~
~~not~~

This doesn't mean that
 $f_n(x) \rightarrow f(x)$ unif. on F ,
just that $|f_n(x) - f(x)| < \delta$
 \forall large n and $\forall x \in F$.

But δ is fixed!

Proof of Egorov:

Given $\varepsilon > 0$.

Step 0: We may assume
that $f_n \rightarrow f$ everywhere.

Step 1: Pick any k .

Then $\exists B_k \subseteq E$

$$\text{s.t. } m(E \setminus B_k) < \frac{\varepsilon}{10^k}$$

$$\text{and } |f_n(x) - f(x)| < \frac{1}{k} \quad \forall x \in B_k, \forall n \geq N_k$$

Let

$$E_1 := \{x: |f_1(x) - f(x)| < 1, \forall n \geq 1\}$$

$$E_2 := \{x: |f_2(x) - f(x)| < 1, \forall n \geq 2\}$$

:

By prior argument, $\exists N_1$

$$\text{s.t. } m(E \setminus E_{N_1}) < \frac{\varepsilon}{10}$$

$$\text{and } |f_n(x) - f(x)| < 1 \quad \forall x \in E_{N_1}, \forall n \geq N_1.$$

Let $B_1 := E_{N_1}$.

Repeat argument but
change 1 to $\frac{1}{2}$.

$$E_1'' := \{x: |f_n(x) - f(x)| < \frac{1}{2} \mid \forall n \geq 1\}$$

$$E_2'' := \{x: |f_n(x) - f(x)| < \frac{1}{2} \mid \forall n \geq 2\}$$

\vdots

$$\text{Then } \exists N_2 \text{ s.t. } m(E \setminus E_{N_2}'') < \frac{\epsilon}{10}$$

$$\text{and } |f_n(x) - f(x)| < \frac{1}{2}$$

$$\forall x \in E_{N_2}'' \quad \forall n \geq N_2$$

$$\text{Let } B_2 := E_{N_2}''$$

More generally, get

N_k s.t.

$$m(E \setminus E_{N_k}^{(k)}) < \frac{\epsilon}{10^k}$$

$$\text{and } |f_n(x) - f(x)| < \frac{1}{k}$$

$$\forall x \in E_{N_k}^{(k)}$$

$$\forall n \geq N_k$$

$$\text{Let } B_k := E_{N_k}^{(k)} //$$

Picture:

$$E_1' \quad E_2' \quad E_3' \quad \dots \quad E_{N_1}' \quad \dots$$

$$\xrightarrow{|f_n(x) - f(x)| < 1}$$

$$E_4'' \quad E_2'' \quad E_3'' \quad \dots \quad E_{N_2}''$$

$$\xrightarrow{|f_n(x) - f(x)| < \frac{1}{2}}$$

...

So we want x 's that
belong to E_{N_1}' and E_{N_2}''
and $E_{N_3}^{(3)}$ and ...

Step 2: Let

$$B := \bigcap_{k=1}^{\infty} B_k$$

$$\text{Then } m(E \setminus B) < \frac{\epsilon}{9}$$

$$\text{and } f_n \rightarrow f \text{ unif. on } B.$$

Pt 2:

$$E \setminus B = \bigcup_{k=1}^{\infty} (E \setminus B_k)$$

$$\Rightarrow m(E \setminus B) \leq \sum_{k=1}^{\infty} m(E \setminus B_k) < \frac{\epsilon}{9}$$

Pick any $\delta > 0$.

Then $\forall k > 1/\delta$,

$$|f_n(x) - f(x)| < \delta \quad \forall x \in B \quad \forall n \geq N_k //$$

Step 3: W.M.

$$\text{Pt 3: We have } m(E \setminus B) < \epsilon/9$$

$$\text{and } f_n \rightarrow f \text{ unif. on } B.$$

Then \exists closed $F \subseteq B$ s.t.

$$m(B \setminus F) < \frac{\epsilon}{10}.$$

Thus,

$$m(E \setminus F) = m((E \setminus B) \cup (B \setminus F))$$

$$\leq \frac{\epsilon}{9} + \frac{\epsilon}{10} < \epsilon$$

$$\text{and } f_n \rightarrow f \text{ unif. on } F.$$

Q.E.D.

Levin's Thm:

Given $f: E \rightarrow \mathbb{R}$ mble
s.t. $m(E) < \infty$. Then
 \exists closed $F \subseteq E$ s.t.
 $m(E \setminus F) < \varepsilon$ and
 $f|_F$ is ct.

Pf: Given $\varepsilon > 0$.

Step 1: Find $f_n: E \rightarrow \mathbb{R}$
s.t.

f_n cts a.e. on E
and $f_n \rightarrow f$ a.e.
we need:

Lemma: Any mble $f: E \rightarrow \mathbb{R}$
is the ptwise a.e. limit
of step functions.

Recall: any step f_n is
defined as follows:
(i) Pick finite collection of
disjoint rectangles R_1, R_2, \dots, R_N

(ii) Set $\varphi(R_k) = \alpha_k$
and $\varphi(x) = 0$ else.

Pf 1: Pick $\{f_n\}$ sequence
of step fns f_n s.t.
 $f_n \rightarrow f$ a.e.

We know f_n is cts. a.e.
and since $f_n \rightarrow f$ a.e.
 \exists mble $E_n \subseteq E$ s.t. f_n is
cts. on E_n and $m(E \setminus E_n) < \frac{\varepsilon}{10^n}$.

Step 2: Find large set

$B \subseteq E$ on which
 $f_n \rightarrow f$ unif.

Pf 2:

Egorov $\Rightarrow \exists$ closed $B \subseteq E$
s.t. $f_n \rightarrow f$ unif. on B
and $m(E \setminus B) < \frac{\varepsilon}{10}$.

Picture:

f_n cts.
everywhere
except E_n^c



On B , $f_n \rightarrow f$ unif.

Step 3: Let $C := B \setminus (\bigcup_{n=1}^{\infty} E_n^c)$

Then f_n is cts. on C
and $f_n \rightarrow f$ unif. on C

Finally,

$$m(C) > m(E) - \frac{\varepsilon}{3}.$$

Now choose large closed
 $F \subseteq C$. Done.

This proves Levin,
except for:

Lemma: Any measurable $f \geq 0$
is the ptwise. a.e.
limit of step f_n .

Note: Can't remove a.e.

But

Propⁿ: Any measurable $f \geq 0$
is the ptwise limit
(everywhere!) of
simple f_n 's.

Any simple $f \geq 0$ is
constructed as follows:

(i) Pick finitely many
disjoint sets E_1, E_2, \dots, E_n
s.t. $m(E_k) < \infty \forall k$.

(ii) $\forall k$, set

$$\psi(E_k) := \alpha_k$$

$$\text{and } \psi(x) = 0 \\ \forall x \notin \bigcup_{k=1}^n E_k.$$