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Last time:

- Proved Egorov's Thm
- Proved Lusin's Thm
  ↳ consequence of Egorov and

Lemma: Any msble f is ptwise a.e. limit of step f^{n}s.

Prop^{n}: Any msble f is

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ptwise limit of simple f^{n}s.

In fact, we can do better if $f \geq 0$:

Prop^{n}: Given msble $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$, then exist simple φ_k s.t. $\varphi_k \nearrow f$. i.e.

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$\varphi_k \to f$ and

$\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots$

Pf: Given msble $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$.

Step 1: Approx. f by a bdd f^{n}, ~~the supp~~ supported on a set of finite measure.

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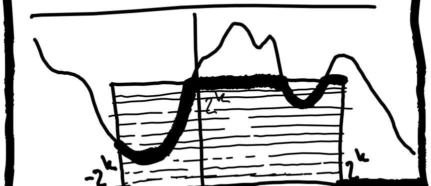
Def^{n}: The support of a f^{n} is

$$\mathrm{supp}(f) := \{x : f(x) \neq 0\}.$$

We say f is supported on A iff $f(x) = 0 \ \forall x \notin A$.



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Our approximation to f will be

$$f_k(t) := \begin{cases} f(t) & \text{if } t \in [-2^k, 2^k] \text{ and } f(t) \leq 2^k \\ 2^k & \text{if } t \in [-2^k, 2^k] \text{ and } f(t) > 2^k \\ 0 & \end{cases}$$

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Step 2: Slice horizontally w/each slice having height $\frac{1}{2^k}$. What's a simple f^{n} φ_k that approximates f_k?

$\varphi_k(t) = 0$ whenever $0 \leq f_k \leq \frac{1}{2^k}$

$\varphi_k(t) = \frac{1}{2^k}$ whenever $\frac{1}{2^k} < f_k \leq \frac{2}{2^k}$

$\varphi_k(t) = \frac{2}{2^k}$ when $\frac{2}{2^k} < f_k \leq \frac{3}{2^k}$

$\vdots$

$\varphi_k(t) = 2^k - \frac{1}{2^k}$ when $2^k - \frac{1}{2^k} < f_k \leq 2^k$

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<u>Note</u>: $0 \le f_k - \varphi_k \le \frac{1}{2^k}$

<u>Step 3</u>: WIN.

We have $f_k \nearrow f$

And φ_k is a simple approx. to f_k from below.

Finally, $\varphi_k \le \varphi_{k+1}$

Thus $\varphi_k \nearrow f$. ∎

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From here, we deduce

<u>Corollary</u>: Given any msble $f: \mathbb{R} \to \mathbb{R}$, $\exists$ simple $\varphi_k \to f$.

<u>Pf</u>: Write $f = f^+ - f^-$

where
$$f^+(t) := \max\{f(t), 0\}$$
$$f^-(t) := \max\{-f(t), 0\}$$

By Prop$^{\text{n}}$, there exist

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simple $\varphi_k^+ \to f^+$

and simple $\varphi_k^- \to f^-$

$$\Rightarrow \underbrace{\varphi_k^+ - \varphi_k^-}_{\text{simple}} \to f^+ - f^- = f$$ ∎

We can say a bit more: Note that one of f^+, f^- is zero @ any input. So $\forall t$, $\varphi_k^+(t) = 0$ or $\varphi_k^-(t) = 0$.

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~~Thus~~ And φ_k^+ is mon. incr.

φ_k^- is mon. incr.

$$\Rightarrow |\varphi_k^+ - \varphi_k^-| = \varphi_k^+ + \varphi_k^-$$

which is mon. incr.

Now if we can approx simple by step fn's, then we'll be able to approx msble fn's by steps!

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<u>Prop$^{\text{n}}$</u>: Any msble f$^{\text{n}}$ is pwise a.e. limit of step f$^{\text{n}}$'s.

<u>Pf</u>: $\exists$ simple f$^{\text{n}}$'s $\varphi_k \to f$.

<u>Step 1</u>: Given any simple f$^{\text{n}}$ φ, $\exists$ step f$^{\text{n}}$ ψ s.t. $\varphi = \psi$ outside of a set of msre $< \varepsilon$.

(b/c finite msble set can be approx. w/in ε by

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finite union of cubes)

<u>Step 2</u>: Thus, $\forall k$, $\exists$ step ψ_k s.t. $\psi_k = \varphi_k$ outside F_k w/ $m(F_k) < \frac{1}{2^k}$.

<u>Step 3</u>: Borel-Cantelli FTW:

For any t, either

(1) $t \in F_k$ for <u>finitely</u> many k
$\Rightarrow \varphi_k(t) = \psi_k(t)$ $\forall$ large k

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$\Rightarrow$

$$|f(t) - \psi_k(t)| \le \varepsilon$$

$$\le \underbrace{|f(t) - \varphi_k(t)|}_{\downarrow\, k\to\infty \atop 0} + \underbrace{|\varphi_k(t) - \psi_k(t)|}_{= \atop 0}$$

$\forall$ large k

(2) $m(\{\text{all other } t\}) = 0$

$\blacksquare$

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