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# Lebesgue Integration

Given $E \subseteq \mathbb{R}^d$ msble.

$$\int_{\mathbb{R}^d} \chi_E := m(E)$$

(We'll ~~__~~ write ~~__~~

$$\int f := \int_{\mathbb{R}^d} f.$$

Some sources write $\int f = \int f(x)\,dx = \int f\,dm = \int^{d} f$ )

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Example:

$$\int \chi_{\mathbb{Q}} = 0 \quad \text{by def}^{\underline{n}}.$$

Def$^{\underline{n}}$: Given finitely many disjoint msble sets $E_1, \dots, E_N \subseteq \mathbb{R}^d$, and $\alpha_1, \dots, \alpha_N \in \mathbb{R}$,

$$\int (\alpha_1 \chi_{E_1} + \dots + \alpha_N \chi_{E_N}) :=$$

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$\alpha_1 m(E_1) + \dots + \alpha_N m(E_N)$.

So we've defined the Lebesgue integral for any simple f$^{\underline{n}}$!

There are several equivalent def$^{\underline{n}}$s of simple f$^{\underline{n}}$s: original def$^{\underline{n}}$ is $f$ is simple iff $f = \alpha_1 \chi_{E_1} + \dots + \alpha_N \chi_{E_N}$ for some disjoint msble sets $E_1, \dots, E_N$.

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Prop$^{\underline{n}}$: ~~f is simple~~

Given msble $f : \mathbb{R}^d \to \mathbb{R}$. The following are equivalent

1. $f$ is simple
2. $f$ can be written $f = \alpha_1 \chi_{E_1} + \dots + \alpha_N \chi_{E_N}$ with ~~__~~ $E_k$'s msble and $\alpha_k$'s $\in \mathbb{R}$
3. $f$ takes on finitely many values.

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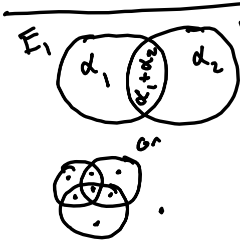
1 $\Rightarrow$ 2 ✓

3 $\Rightarrow$ 1 exercise.

Proof by pictures that 2 $\Rightarrow$ 3




4 diff. values

8 diff. values.

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~~__~~ To make this rigorous: $\forall i$, $x \in E_i$ or $x \in E_i^c$.

Prop$^{\underline{n}}$: $\int (\alpha_1 \chi_{A_1} + \dots + \alpha_M \chi_{A_M}) = \alpha_1 m(A_1) + \dots + \alpha_M m(A_M)$.

~~Proof~~

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Proof: From above, suffices to show that if $\alpha_1 \chi_{A_1} + \cdots + \alpha_M \chi_{A_M} =$

$$= \beta_1 \chi_{B_1} + \cdots + \beta_N \chi_{B_N}$$

then $\sum_k \alpha_k m(A_k) = \sum_k \beta_k m(B_k)$

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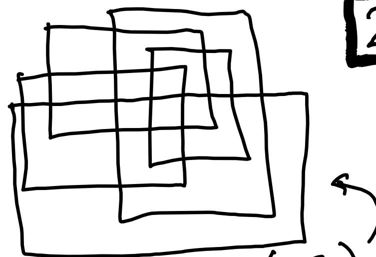

Consider $\left(\bigcup A_k\right) \cup \left(\bigcup B_k\right)$

Claim (left as exercise) that can split this into finitely many disjoint msble sets.

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More precisely, $\exists$ msble disjoint $E_1, E_2, \dots, E_L$ s.t. $\mathbb{R}^d = \bigsqcup_{\ell=1}^{L} E_\ell$ and $\forall k$,

$$A_k = \bigsqcup_{\ell \in S_k} E_\ell$$

$$B_k = \bigsqcup_{\ell \in T_k} E_\ell$$

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we have

$$\sum_k \alpha_k m(A_k) = \sum_k \alpha_k m\Big(\bigsqcup_{\ell \in S_k} E_\ell\Big)$$

$$= \sum_k \alpha_k \sum_{\ell \in S_k} m(E_\ell)$$

$$= \sum_k \alpha_k \sum_\ell \chi_{S_k}(\ell) m(E_\ell)$$

$$= \sum_\ell m(E_\ell) \sum_k \alpha_k \chi_{S_k}(\ell)$$

$$= \sum_\ell \Big( \sum_k \alpha_k \chi_{S_k}(\ell) \Big) m(E_\ell)$$

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Similarly,

$$\sum_k \beta_k m(B_k) = \sum_\ell \Big( \sum_k \beta_k \chi_{T_k}(\ell) \Big) \cdot m(E_\ell)$$

So E.T.S. $\forall \ell$,

$$\sum_k \alpha_k \chi_{S_k}(\ell) = \sum_k \beta_k \chi_{T_k}(\ell)$$

Recall that $\forall t$,

$$\sum_k \alpha_k \chi_{A_k}(t) = \sum_k \beta_k \chi_{B_k}(t)$$

Pick $t \in E_4$.

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Does $t \in A_1$?

$t \in A_1 \iff 4 \in S_1$

$t \in A_2 \iff 4 \in S_2$

So

$$t \in E_4 \Rightarrow \sum_k \alpha_k \chi_{A_k}(t) = \sum_k \alpha_k \chi_{S_k}(4)$$

Similarly

$$t \in E_4 \Rightarrow \sum_k \beta_k \chi_{B_k}(t) = \sum_k \beta_k \chi_{T_k}(4)$$

$$\Rightarrow \sum_k \alpha_k \chi_{S_k}(4) = \sum_k \beta_k \chi_{T_k}(4)$$

∎

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Thus, we've defined integration for simple fn's over all $\mathbb{R}^d$. What about integral of simple fn $f$ over subset $A \subseteq \mathbb{R}^d$?

$$\int_A f := \int_{\mathbb{R}^d} f \cdot \chi_A .$$

(A must be msble)

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Expect: given simple $f, g$

① $\int \alpha f + \beta g = \alpha \int f + \beta \int g$

② $\int_{A \sqcup B} f = \int_A f + \int_B f$

③ $f \leq g \Rightarrow \int f \leq \int g$

④ $\left| \int f \right| \leq \int |f|$.

⑤ If $f = g$ a.e. then $\int f = \int g$.

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Recall when proving that any msble $f$ is limit of simple fn's;

$f \rightsquigarrow f^+ - f^-$

$f \geq 0 \longrightarrow f_k$ · bdd · supported on set of finite msre

We'll use the same reductive steps to define

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the Lebesgue integral for any msble $f$.

This integral will have nice properties, e.g. if $\{f_n\}$ msble and $|f_n| \leq g$ ("nice")

$\Rightarrow \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$.

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