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Last time:

we developed a theory of integration for simple fns:

$$\int \sum_{n\le N} \alpha_n \chi_{E_n} = \sum_{n\le N} \alpha_n m(E_n)$$

We proved that if simple $f = g$ a.e., then $\int f = \int g$.

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Recall from our proof that any msble f is the limit of simple fns, we reduced the problem:

$$f \rightsquigarrow f = f^+ - f^- \quad \text{w/ } f^\pm \ge 0.$$

$$g \ge 0 \rightsquigarrow \exists\, g_k \nearrow g,$$

w/ $g_k = \begin{cases} \cdot \text{ bounded} \\ \cdot \text{ supported on set of finite msre} \end{cases}$

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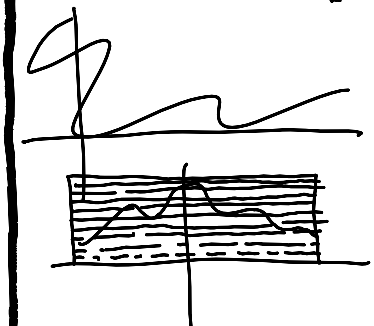
$h =$ BS fn $\rightsquigarrow$ $\exists$ simple fns $\psi_k \approx h$



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Integrating BS

Given f a BS fn, say $|f| \le M$ and $\operatorname{supp} f \subseteq E$.

Then $\exists$ simple $\psi_k \to f$ a.e. s.t. $|\psi_k| \le M$ and $\operatorname{supp} \psi_k \subseteq E$.

How should we define

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$\int f$?

Def$^{\text{n}}$: $\int f := \lim_{k\to\infty} \int \psi_k$.

We have to check that ② any such choice of ψ_k yields same value for $\int f$, and that

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for any such ψ_k,

① $\lim_{k\to\infty} \int \psi_k$ exists.

Proof of ①:

Given $\psi_k \to f$ a.e. s.t. $|\psi_k| \le M$ and $\operatorname{supp} \psi_k \subseteq E$.

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Egorov $\Rightarrow$

$\psi_k \longrightarrow f$ uniformly on some huge $F \subseteq E$.

$$\left| \int \psi_m - \int \psi_n \right| = \left| \int_E \psi_m - \int_E \psi_n \right|$$

$$= \left| \int_E \psi_m - \psi_n \right|$$

$$\le \int_E |\psi_m - \psi_n| =$$

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$$= \int_{\underbrace{F}_{\text{finite msre}}} \underbrace{|\psi_m - \psi_n|}_{\text{tiny}} + \int_{\underbrace{E \setminus F}_{\text{tiny}}} \underbrace{|\psi_m - \psi_n|}_{\text{bdd}}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, the sequence $\int \psi_k$ is Cauchy. ∎

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<u>Proof of ②</u>:

Given two simple sequences φ_k and ψ_k, s.t.

$\varphi_k \longrightarrow f$ a.e.

$\psi_k \longrightarrow f$ a.e.

$|\varphi_k| \le M$, $|\psi_k| \le M$,

supp φ_k, supp $\psi_k \subseteq E$.

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<u>Know</u>: $\varphi_k - \psi_k \longrightarrow 0$ a.e.

<u>Want</u>: $\int \varphi_k - \psi_k \longrightarrow 0$.

Given simple ξ_k, BS s.t. $\xi_k \longrightarrow 0$ a.e.

Egorov $\Rightarrow$ $\exists$ huge $F \subseteq E$ s.t. $\xi_k \longrightarrow 0$ unif. on F.

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$\Rightarrow$

$$\left| \int_E \xi_k \right| \le \int_E |\xi_k|$$

$$= \int_{\underbrace{F}_{\text{bdd}}} \underbrace{|\xi_k|}_{\text{tiny}} + \int_{\underbrace{E \setminus F}_{\text{tiny}}} \underbrace{|\xi_k|}_{\text{bdd}}$$

$< \varepsilon$ $\forall$ large k.

Thus $\int_E \xi_k \longrightarrow 0$. ∎

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We just proved

<u>if</u> simple BS $\xi_k \longrightarrow 0$ a.e.

<u>then</u> $\int \xi_k \longrightarrow 0$.

<u>Prop^{n}</u>: If $\int f = 0$ for some BS $f \ge 0$, then $f = 0$ a.e.

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Prop^n: If $\int f = 0$ (for f = BS ≥ 0), then $f = 0$ a.e.

Proof idea:

$$0 = \int f = \int_{\{f=0\}} \cancel{f} + \int_{\{f>0\}} f$$

Let $A_n := \{x : f(x) > \frac{1}{n}\}$

If $f(x) > 0$, then $x \in \infty$ly many A_n

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Borel-Cantelli

$\Rightarrow$ $f(x) > 0$ on a set of msre 0. ■

Need to know that

$$\sum_{n=1}^{\infty} m(A_n) < \infty.$$

If $m(A_k) > 0$, then

$\int f > \frac{1}{k} m(A_k)$ since

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$$\frac{1}{k}\chi_{A_k}(t) = \begin{cases} \frac{1}{k} & \text{if } f(t) > \frac{1}{k} \\ 0 & \text{———} \end{cases}$$

$$\leq f(t)$$

$$\Rightarrow \underbrace{\int \frac{1}{k}\chi_{A_k}}_{= \frac{1}{k} m(A_k)} \leq \int f = 0 \quad \forall k$$

$\Rightarrow m(A_k) = 0 \quad \forall k$

$\Rightarrow m(\{f > \frac{1}{k}\}) = 0 \quad \forall k$

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$$\Rightarrow m(\{f > 0\}) = m\left(\bigcup_{k=1}^{\infty} \{f > \tfrac{1}{k}\}\right)$$

$$\leq \sum_{k=1}^{\infty} m(\{f > \tfrac{1}{k}\})$$

$$= 0$$

■

Implicit in proof: $\int$ BS satisfies expected properties:

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Prop^n: $\forall f, g$ BS f^{n}s,

① $\int \alpha f + \beta g = \alpha \int f + \beta \int g$

② $\int_{A \sqcup B} f = \int_A f + \int_B f$

③ $f \leq g \Rightarrow \int f \leq \int g$

④ $\left|\int f\right| \leq \int |f|$.

All inherited from the same property for $\int \varphi_k$.

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One big advantage of Lebesgue integrals is that they play nice w/ LEOs.

Cautionary example:

Let $\varphi_n(x) := \begin{cases} n & \text{on } (0, \frac{1}{n}) \\ 0 & \text{on } [\frac{1}{n}, 1] \end{cases}$

Note: $\forall n$, φ_n = simple, BS.

$\int \lim_{n\to\infty} \varphi_n = \int 0 = 0$

$\lim \int \varphi_n = \lim 1 = 1$.

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Nonetheless:

Bounded Convergence Thm:

Given any sequence $\{f_n\}$ msble s.t. $\forall n$

- $|f_n| \le M$ and
- $\text{supp } f_n \subseteq E$ of finite msre.

If $f_n \to f$ a.e., then $\int f_n \to \int f$.

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Pf: Egorov $\Rightarrow \exists$ large $F \subseteq E$ s.t. $f_n \to f$ unif on F

$$\Rightarrow \left| \int_E f_n - f \right| \le \int_E |f_n - f|$$

$$= \int_{\underbrace{F}_{\text{bdd}}} \underbrace{|f_n - f|}_{\text{tiny}} + \int_{\underbrace{E \setminus F}_{\text{tiny}}} \underbrace{|f_n - f|}_{\text{bdd}}$$

$$< \varepsilon.$$ ∎

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Note: in all our Egorov proofs, we prove something stronger than the claim:

$$\int |f_n - f| \to 0.$$

This is a stronger notion of convergence that a.e.: if this holds, we say $f_n \to f$ in L^1.

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We now use BCT to prove that Riemann integration is subsumed by Lebesgue integration.

Recall given $f : [a,b] \to \mathbb{R}$, the lower Riemann sum is

$L(f, P)$

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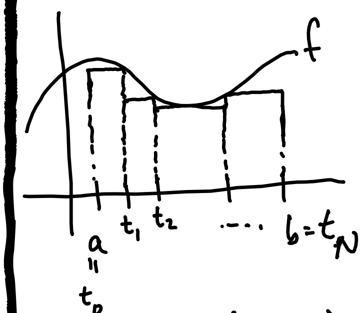

$$L(f, P) = \sum_{k \le N} (t_k - t_{k-1}) m_k$$

where $m_k := \inf f([t_{k-1}, t_k])$

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Note: $L(f, P) = \int \text{step } f$

Thm: If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then

$$\underbrace{\int_a^b f}_{R} = \underbrace{\int_{[a,b]} f}_{L}$$

Pf: By Lebesgue criterion, $f =$ cts. a.e. $\Rightarrow f$ msble.

So $\exists \psi_k$ step s.t. $\psi_k \to f$ a.e.

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$$\int_a^b \psi_k = \int_{[a,b]} \psi_k .$$

by def^{n}

Thus

$$\int_a^b f = \lim_{k\to\infty} \int_a^b \psi_k = \lim_{k\to\infty} \int_{[a,b]} \psi_k$$

by BCT

$$= \int_{[a,b]} \lim_{k\to\infty} \psi_k$$

$$= \int_{[a,b]} f \qquad \blacksquare$$

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