1

We've developed a theory of integration for simple f^{n}s

$$\int \sum_{n\le N} \alpha_n \chi_{E_n} := \sum \alpha_n m(E_n)$$

- BS f^{n}s:

$$\int f := \lim_{k\to\infty} \int \psi_k \quad \text{where}$$

simple $\psi_k \longrightarrow f$ a.e. and $|\psi_k| \le M$ and $\mathrm{supp}(\psi_k) \subseteq E$.

2

We proved the Bdd. Convergence Thm:

$\{f_n\}$ msble BS s.t. $|f_n| \le M$, $\mathrm{supp}(f_n) \subseteq E$

Then $f_n \longrightarrow f$ a.e.

$$\Rightarrow \int f_n \longrightarrow \int f$$

(This is a LEO)

3

We deduced that Lebesgue integration subsumes Riemann integration.

Next step: ~~set up~~ define $\int f$ for any msble $f \ge 0$.

Natural guess:

4

Create a sequence of BS f^{n}s $f_n \longrightarrow f$

and define $\int f := \lim_{n\to\infty} \int f_n$

Q: Can you ~~give~~ an example where

BS $f_n \longrightarrow f$

and BS $g_n \longrightarrow f$

but $\lim \int f_n \ne \lim \int g_n$

5

Recall travelling salesman

$$\sigma_n = \chi_{[n,n+1]} \longrightarrow 0$$

but $\lim_{n\to\infty} \int \sigma_n = \lim_{n\to\infty} 1 = 1$

$$\ne \lim_{n\to\infty} \int 0$$

Instead:

Def^n: Given msble ~~[illegible]~~ $f: \mathbb{R}^d \longrightarrow [0,\infty]$, let

$$\int f := \sup \left\{ \int g : \text{BS } g \text{ s.t. } 0 \le g \le f \right\}$$

6

Let $f_n := \frac{1}{n} \chi_{[0,n]}$

$\downarrow$ unif.

0

but $\int f_n = 1 \quad \forall n$.

Def^n: $\int_E f := \int f \cdot \chi_E$.

This satisfies all usual properties (linearity, additivity, monotonicity) ...

1

Another nice prop. we've proved before is

$$\int f = 0 \Rightarrow f = 0 \text{ a.e.}$$

Similarly,

<u>Prop$^{\underline{n}}$</u>: $\int f < \infty \Rightarrow f < \infty$ a.e.

<u>Fundamental LEOs</u>

We've seen a BS LEO in the BCT.

2

What about LEOs for msble $f \geq 0$?

We know in general ~~this~~ $\int \lim \neq \lim \int$:

1) $\sigma_n := \chi_{[n,n+1]}$

2) $\tau_n := \frac{1}{n}\chi_{[0,n]}$

3) $\varphi_n := n\,\chi_{[0,\frac{1}{n}]}$

In all these cases, $\int \lim = 0 \neq 1 = \lim \int$.

3

In all these examples, mass escapes to ∞.

Note in all three of these, $\int \lim \leq \lim \int$.

<u>Lemma</u>: Given $f_n \geq 0$ msble

Then $\int \lim_{n\to\infty} f_n \leq \lim_{n\to\infty} \int f_n$.

assuming limits exist.

Tao: "in the limit, mass can be destroyed but not created."

4

<u>Idea</u>:

Let $f := \lim_{n\to\infty} f_n$.

By def$^{\underline{n}}$:

$$\int \lim f_n = \int f = \sup_{\substack{g \text{ BS} \\ 0 \leq g \leq f}} \int g$$

Thus E.T.S. for any BS g s.t. $0 \leq g \leq f$,

$$\int g \leq \lim \int f_n$$

5

Know a LEO for BS f$^{\underline{n}}$s:

$$\text{BCT} \Rightarrow \int g = \lim_{n\to\infty} \int g_n$$

when $g_n \to g$ and g_n BS.

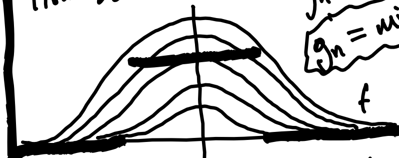
So if we can choose $g_n \to g$ BS s.t. $g_n \leq f_n \ \forall n$, we win:

6

$$\int g = \int \lim g_n = \lim \int g_n \leq \lim \int f_n$$

How do we choose g_n's?

$g_n = \min(f_n, g)$?



f = top f$^{\underline{n}}$ $\quad f_n \to f$ from below

g in bold.

<u>Want</u>: $g_n \to g$, $g_n \leq f_n \ \forall n$

1

<u>Lemma</u>: $\int \lim f_n \le \lim \int f_n$

when $f_n \ge 0$ msble.

<u>Pf</u>: Let g be an arbitrary BS f^n s.t. $0 \le g \le \lim_{n\to\infty} f_n$. Say $|g| \le M$ and $\operatorname{supp}(g) \subseteq E$.

Set $g_n := \min\{g, f_n\}$: note $|g_n| \le M$ and

$\operatorname{supp}(g_n) \subseteq E$

Moreover, $\lim_{n\to\infty} g_n = \min\{g, \lim f_n\}$

$= g$.

2

BCT $\Rightarrow$

$$\int g = \int \lim g_n = \lim \int g_n$$

$$\le \lim \int f_n$$

Since g was arbitrary, $\int \lim f_n := \sup \int g \le \lim \int f_n$. ■

What if $\lim_{n\to\infty} f_n$ doesn't exist? Turns out we can prove a similar

3

theorem <u>even</u> if $\lim_{n\to\infty} f_n$ doesn't exist.

To see ~~the~~ how, ...

Consider $a_n := \sin(n)$.

$\lim a_n$ d.n.e. but can still say something about long term behavior of a_n:

$\sup a_n = 1$

$\inf a_n = -1$

<u>Note</u>: $a_n \ne \pm 1$

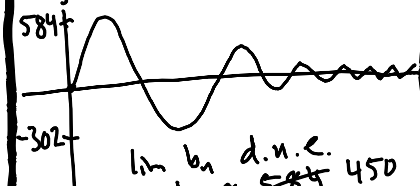
4

Instead consider

$$b_n := \frac{n+1000}{n+1}\sin(2n).$$

$b_1 \approx 450$ $b_2 \approx -250$

$b_3 \approx -70$



$\lim b_n$ d.n.e.

$\sup b_n \approx \cancel{584}\ 450$

$\inf b_n \approx \cancel{-302}\ -250$

5

Neither sup nor inf tells you much about asymptotic behavior of b_n. What can you say?

The eventual sup is 1

the eventual inf is -1.

i.e. $\lim_{N\to\infty} \sup\{b_n : n \ge N\}$

$=: \limsup b_n$

$\liminf b_n := \lim_{N\to\infty} \inf\{b_n : n \ge N\}$

6

Our proof of our lemma produces a stronger result:

<u>Fatou's Lemma</u>:

Given msble $f_n \ge 0$.

Then $\int \liminf_{n\to\infty} f_n \le$

$\le \liminf_{n\to\infty} \int f_n$