1

Last time, failed to prove a general LEO for msble $f \geq 0$, b/c mass can escape to ∞.

However, we proved a \LEqO :

<u>Fatou's Lemma</u>: Given $f_n \geq 0$ msble. Then $\int \liminf f_n \leq \liminf \int f_n$

2

Amazingly, can use Fatou to derive a LEO:

<u>Monotone Convergence Thm</u>: Given $f_n \geq 0$ msble, s.t. $f_n \nearrow f$. Then $\int f_n \to \int f$.

Here, $f_n \nearrow f$ means $f_n \to f$ a.e. and $f_n \leq f_{n+1}$ a.e.

3

<u>Pf</u>: $f_n \leq f$ a.e. $\forall n$

$\Rightarrow \int f_n \leq \int f$

$\Rightarrow \limsup_{n\to\infty} \int f_n \leq \int f$

But also,

$$\int f = \int \lim_{n\to\infty} f_n = \int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n \quad \text{by Fatou}$$

Thus:

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n$$

4

Thus

$\lim_{n\to\infty} \int f_n$ exists and $= \int f.$ ∎

<u>Corollary à la Cauchy</u>:

Given $f_n \geq 0$ msble,

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n .$$

5

<u>Pf</u>: Let

$F_N := \sum_{n\leq N} f_n$ and $F := \sum_{n=1}^{\infty} f_n$.

Thus, $F_N \nearrow F$, so

MCT $\Rightarrow$

$$\int \sum_{n=1}^{\infty} f_n = \int F = \lim_{N\to\infty} \int F_N = \lim_{N\to\infty} \int \sum_{n\leq N} f_n = \lim_{N\to\infty} \sum_{n\leq N} \int f_n = \sum_{n=1}^{\infty} \int f_n .$$

∎

6

<u>Corollary (Borel-Cantelli)</u>:

If $\sum_{n=1}^{\infty} m(E_n) < \infty$, then

$m(\{x : x \in E_n \text{ for infly many } n\}) = 0.$

<u>Pf</u>: $\int \sum \chi_{E_n} = \sum_{n=1}^{\infty} \int \chi_{E_n} = \sum_{n=1}^{\infty} m(E_n) < \infty$

$\Rightarrow \sum_{n=1}^{\infty} \chi_{E_n} < \infty$ a.e.

∎

1

S'pose $f \geq 0$ msble and $\int f < \infty$. We know that $f < \infty$ a.e.

Q: what else should be true about f?

Conj: $\lim_{x \to \pm\infty} f(x) = 0$.

More generally, as $|x| \to \infty$, $f \to 0$.

False: on HW.

2

~~f must~~

Conj 2: f must be small most everywhere.

Conj 3: $\exists$ large sets on which f is bdd.

Conj 4: $|f|^2 < \infty \ldots$

Prop$^{\underline{n}}$: S'pose $f : \mathbb{R}^d \to [0,\infty]$ is msble and $\int f < \infty$. Then $\exists r$ s.t. $\int_{\{|x|>r\}} f(x) < \varepsilon$.

3

Pf idea: Want to find r s.t.

$$\int f - \int_{B_r(0)} f < \varepsilon.$$

Consider

$$\int f - \int_{B_1(0)} f, \quad \int f - \int_{B_2(0)} f,$$

$$\int f - \int_{B_3(0)} f, \quad \ldots$$

4

Want this to tend to 0. i.e. want to study the sequence

$$\int_{B_1(0)} f \leq \int_{B_2(0)} f \leq \int_{B_3(0)} f \leq \cdots$$

Can't directly apply MCT; but instead:

$$f \cdot \chi_{B_n(0)}$$

5

Proof: Given $\varepsilon > 0$.

Let $f_n := f \cdot \chi_{B_n(0)}$.

Then $f_n \nearrow f$, so MCT

$\Rightarrow \int f_n \longrightarrow \int f$

$\Rightarrow \exists N$ s.t. $\int f - \int f_n < \varepsilon$. $\blacksquare$

$$\int f - \int_{B_n(0)} f = \int_{B_n(0)^c} f$$

6

In this proof, we defined f_n to grow w/ the domain: $f_n := f \cdot \chi_{\{|x| \leq n\}}$

What if we set up a sequence that grows w/ the range? i.e.

$$g_n := f \cdot \chi_{\{f \leq n\}} \quad ?$$

Then $g_n \nearrow f \overset{MCT}{\Longrightarrow} \int g_n \longrightarrow \int f$

Thus, $\exists N$ s.t. $\int f - g_N$ is tiny.

1

This observation yields a beautiful property of Lebesgue integral: it's "continuous." i.e.

<u>Thm</u>: Given msble $f:\mathbb{R}^d \to [0,\infty]$ s.t. $\int f < \infty$. Then $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$$m(E) < \delta \Rightarrow \int_E f < \varepsilon.$$

2

<u>Proof</u>: Given $\varepsilon > 0$.

Let $g_n := f \cdot \chi_{\{f \leq n\}}$.

From above, $\exists N$ s.t.

$$\int f - g_N < \frac{\varepsilon}{2}.$$

Note that $\int_E g_N = \int_E f \cdot \chi_{\{f \leq N\}}$

$$\leq m(E) \cdot N$$

3

Thus,

$$m(E) < \frac{\varepsilon}{2N} \Rightarrow \delta$$

$$\Rightarrow \int_E f = \int_E (f - g_N) + \int_E g_N$$

$$\leq \int (f - g_N) + N \cdot m(E)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \blacksquare$$

4

Integrating general measurable $f^{\underline{n}}$s

Given $f: \mathbb{R}^d \longrightarrow [-\infty, \infty]$ msble, we can write

$$f = f^+ - f^- \quad \text{where}$$

$$f^+ := \max\{f, 0\} \quad \text{and}$$

$$f^- := \max\{-f, 0\}.$$

<u>Note</u>: $f^{\pm}$ are msble, ≥ 0.

5

Then we define

$$\int f := \int f^+ - \int f^-.$$

<u>Issue</u>: What if get $\infty - \infty$?!

<u>Def$^{\underline{n}}$</u>: We say $f: \mathbb{R}^d \to [-\infty, \infty]$ is "<u>integrable</u>" iff $\int |f| < \infty$.

<u>Note</u>: $f^{\pm} \leq |f|$

6

<u>Prop$^{\underline{n}}$</u>: If f is intgble and $f = f_1 - f_2$ for some integble $f_1, f_2 \geq 0$, then

$$\int f = \int f_1 - \int f_2.$$

Dominated Convergence Thm (Lebesgue)

1

Dominated Convergence Thm

S'pose $f_n \longrightarrow f$ a.e. s.t. f_n msble and $|f_n| \leq g$ and g is integrable. Then $\int f_n \longrightarrow \int f$.

2

Colloquially:

If $\{f_n\}$ converges and is dominated by some integrable fn, then $\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$.

3

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