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Last time: we completed a theory of integration. We proved

Prop^{n} 1: Given f integrable. $\forall \varepsilon > 0$ $\exists r$ s.t. $\int_{B_r(0)^c} f < \varepsilon$

Prop^{n} 2: Given f integrable. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $m(E) < \delta \Rightarrow \int_E f < \varepsilon$.

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In both proofs, we constructed sequence $\to f$ and applied MCT:

- in Prop^{n} 1: $f_n := f \cdot \chi_{\{|x| \le n\}}$
- in Prop^{n} 2: $g_n := f \cdot \chi_{\{f \le n\}}$.

Now we'll use these ideas to prove

Dominated Convergence Thm:

Given msble f_n s.t. $|f_n| \le g$

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where g is intgble. If $\lim_{n\to\infty} f_n$ exists, then

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n.$$

Pf: Let $f := \lim_{n\to\infty} f_n$.

E.T.S. $f_n \to f$ in L^1, i.e. $\int |f_n - f| \to 0$.

Now for any set E,

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$$\int |f_n - f| = \int_E |f_n - f| + \int_{E^c} |f_n - f|$$

$$\le \int_E |f_n - f| + \int_{E^c} |f_n| + \int_{E^c} |f|$$

$$\le \int_E |f_n - f| + 2\int_{E^c} g$$

Want this to be small $\forall$ large n. If choose E

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s.t. $E \supseteq B_r(0)$ w/ r = huge, then

$$2\int_{E^c} g < \frac{\varepsilon}{2}.$$

So want to choose such an E s.t. we also have

$$\int_E |f_n - f| < \frac{\varepsilon}{2}.$$

i.e. want to choose E

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s.t. $f_n \cdot \chi_E \to f \cdot \chi_E$ in L^1.

This looks like BCT:

if uniformly BS sequence $g_n \to g$ a.e. then $g_n \to g$ in L^1.

Now choose

$$E_r := \{x : |x| \le r \text{ and } g \le r\}.$$

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Same argument as in Prop 1 $\Rightarrow \exists r$ s.t.

$$\int_{E_r^c} g < \frac{\varepsilon}{4}.$$

And: $f_n \cdot \chi_{E_r} \longrightarrow f \cdot \chi_{E_r}$

$|f_n \chi_{E_r}| \leq r \quad \forall n$

$\mathrm{supp}(f_n \cdot \chi_{E_r}) \subseteq E_r. \; \forall n$

BCT $\Rightarrow f_n \chi_{E_r} \longrightarrow f \cdot \chi_{E_r}$

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in L^1

$\Rightarrow \forall$ large n, $\int_{E_r} |f_n - f| < \frac{\varepsilon}{2}.$ ∎

The space L^1:

Let $L^1 := \{\text{integrable } f \text{ on } \mathbb{R}^d\}$

What structure does L^1 have?

Algebraic structure: L^1

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is a vect. sp. over $\mathbb{R}$.

Analytic structure:

Is there a metric on L^1? i.e., a way to measure distance between two functions?

Set $d(f,g) := \int |f-g|$

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For this to be a metric, we need:

① $d(f,g) = 0$ iff $f = g$

② $d(f,g) = d(g,f)$

③ $d(f,g) + d(g,h) \geq d(f,h)$

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① fails; e.g. $d(\chi_{\mathbb{Q}}, 0) = 0$.

② ✓ ③ ✓

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Note: ① only fails for f, g w/ $f = g$ a.e.

So instead, we define

$L^1 := \{\text{integrable } f \text{ on } \mathbb{R}^d, \text{ up to equivalence a.e.}\}$

i.e. $f \sim g$ iff $f = g$ a.e.

So now L^1 is a vect. sp. and has a

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metric.

Note: this also gives us a way to measure the size of a single $f^{\underline{n}}$:

$\|f\|_{L^1} := d(f,0) = \int |f|$

"L^1-norm of f"

This is sort-of like measuring length of a vector: $\|f\| \geq 0$, $\|f\| = 0$ iff $f = 0$

$\|f+g\| \leq \|f\| + \|g\|$

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A natural question: is L^1 complete w.r.t. this metric?

Informally: a space is complete iff you never leave the space via a convergent sequence.

e.g. $\mathbb{R}$ complete, $\mathbb{Q}$ isn't.

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Formally: a space is complete iff every Cauchy sequence converges to a point in that space.

Thm (Riesz-Fischer): L^1 is complete.

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Proof: Given Cauchy $\{f_n\}$, i.e. $\|f_m - f_n\| \to 0$ as $m,n \to \infty$.

Step 1: Construct a subsequence $f_{n_k} \longrightarrow f \in L^1$. exists!

Step 2: $f_n \longrightarrow f$ in L^1.

Pf of Step 2: By

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Step 1, we know

$$\|f_{n_k} - f\| < \frac{\varepsilon}{2} \quad \forall \text{large } k$$

Since f_n is Cauchy, $\forall$ large n and large k, $\|f_n - f_{n_k}\| < \frac{\varepsilon}{2}$.

$$\Rightarrow \|f_n - f\| < \varepsilon \quad \forall \text{large } n.$$

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Recall from PS 8:

$$\int f_n \longrightarrow 0 \not\Rightarrow f_n \longrightarrow 0_{\text{a.e.}}$$

Even if you require $f_n \geq 0$, still false!

But we'll prove below that $f_n \to 0$ in L^1

$$\Rightarrow \exists f_{n_k} \text{ s.t. } f_{n_k} \longrightarrow 0 \text{ a.e.}$$

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Proof of Step 1:

(f_n) Cauchy $\Rightarrow$

$\exists$ subseq. (f_{n_k}) s.t.

$$\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k} \quad \forall k.$$

Want to write down limit of f_{n_k} as a sum, for example?

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Let

$$f := f_{n_1} + \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})$$

Note: partial sums are f_{n_k}.

~~We~~ Does this converge? We have

$$|f| \leq |f_{n_1}| + \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$$

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$$\Rightarrow \int |f| \leq \int |f_{n_1}| + \int \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$$

$$= \|f_{n_1}\| + \sum_{k=1}^{\infty} \underbrace{\int |f_{n_{k+1}} - f_{n_k}|}_{\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k}}$$

$< \infty$.

Thus $f \in L^1$.

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It follows that

$$f_{n_k} \longrightarrow f \quad \text{a.e.}$$

Since f_{n_k} are the partial sums of f.

However, this isn't what we're trying to prove: we want to show that $f_{n_k} \longrightarrow f$ in L^1.

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Even though we stated Dominated Convergence Thm as a LEO, what we actually proved is:

$$\boxed{\begin{array}{l} \text{If } f_n \longrightarrow f \text{ a.e.} \\ \text{and } |f_n| \leq g \in L^1, \\ \text{then } f_n \longrightarrow f \text{ in } L^1. \end{array}}$$

Thus if we can

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show $|f_{n_k}|$ is bdd by some integrable fn, we'll be done w/ the proof by applying the DCT! Observe: $\forall j$,

$$|f_{n_j}| = \left| f_{n_1} + \sum_{k \leq j-1} (f_{n_{k+1}} - f_{n_k}) \right|$$

$$\leq |f_{n_1}| + \sum_{k \leq j-1} |f_{n_{k+1}} - f_{n_k}|$$

$$\leq |f_{n_1}| + \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}| \in L^1 \quad \blacksquare$$