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Last time:

$L^1 := \{f \text{ integrable on } \mathbb{R}^d\}/\sim$

where $f \sim g$ iff $f = g$ a.e.

L^1 is a vect. sp. w/a norm $\|f\|_{L^1} := \int |f|$

and thus a metric

$d(f,g) := \|f-g\|_{L^1}$.

2

We discussed convergence w.r.t. this metric:

$f_n \to f$ in L^1 iff

$\|f_n - f\|_{L^1} \to 0$.

iff $\int |f_n - f| \to 0$.

In this language:

DCT: Given msble $f_n \to f$ a.e. s.t. $|f_n| \le g \in L^1$. Then

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$f_n \to f$ in L^1.

BCT is similar.

By contrast, MCT is an honest LEO.

(MCT is also called Beppo Levi's Lemma.)

We proved (most of)

Thm (Riesz-Fischer):

L^1 is complete.

4

L^1 is a huge space. To understand it better, think about how we think about $\mathbb{R}$.

How big is π?

$3 < \pi < 4$

$3.13 < \pi < 3.15$

$3.1415926535 < \pi < \cdots$

We understand reals in terms of rationals.

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This works well b/c $\mathbb{Q}$ dense in $\mathbb{R}$. Similarly,

Def$^{\underline{n}}$: We say $\mathcal{F} \subseteq L^1$ is dense in L^1 iff $\forall f \in L^1$ $\forall \varepsilon > 0$ $\exists \varphi \in \mathcal{F}$ s.t.

$\|\varphi - f\|_{L^1} < \varepsilon$.

Examples of dense subsets of L^1:

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1. $\{$All simple $f^{\underline{n}}$s$\}$
2. ? $\{$all polynomials$\}$?
3. $\{$all BS $f^{\underline{n}}$s$\}$
4. $\{$cts $f^{\underline{n}}$s$\}$
5. $\{$step $f^{\underline{n}}$s$\}$

0. L^1

1

Have a geometric intuition for L^1 b/c can measure "length" of any $f \in L^1$. What about angles between $f, g \in L^1$? How do we measure angle between $(1,2,3)$ and $(1,1,2)$? Dot products!

2

Recall:

$$\vec{v} \cdot \vec{w} = \sum v_i w_i = |\vec{v}| \cdot |\vec{w}| \cdot \cos\theta$$

So if we had a dot product on L^1, we could measure "angle". How to define such a dot product? We can think of any $\vec{v} \in \mathbb{R}^3$

3

as a $f^{\underline{n}} : \{1,2,3\} \to \mathbb{R}$

e.g. $(2,7,-5)$ would be

$$1 \mapsto 2$$
$$2 \mapsto 7$$
$$3 \mapsto -5$$

So $f \cdot g = \sum_i f(i) g(i)$

What about $f, g \in L^1$?

$\langle f, g \rangle$ $\cancel{f \cdot g}$ $:= \int fg$

But there's a problem:

4

fg might not $\in L^1$.

e.g. let $f(t) := \frac{1}{\sqrt{t}} \chi_{(0,1]}(t)$.

Then $\int f = 2\sqrt{t}\Big|_0^1 = 2$

but $|\langle f, f \rangle| = \left|\int f^2\right| = \left|\int_0^1 \frac{1}{t}\right| = \infty$

So <u>Cheap fix</u>:

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Let $L^2 := \{f : \int f^2 < \infty\}/\sim$

Turns out, $\forall f, g \in L^2$, $|\langle f, g \rangle| < \infty$.

So now we have a space where we can measure angles between $f^{\underline{n}}$s! But there's a problem: our norm $\|f\|_{L^1}$ might be ∞!

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e.g. $f(t) := \frac{1}{t} \chi_{[1,\infty)}(t) \in L^2$

but $\|f\|_{L^1} = \infty$.

$\Rightarrow f \notin L^1$.

<u>Conclusions</u>:

1. $L^2 \not\subseteq L^1$ and $L^1 \not\subseteq L^2$.
2. We need a different norm/metric on L^2.

1

Fortunately: we can measure length using dot products! Recall:

$$\vec{v}\cdot\vec{v} = v_1^2+v_2^2+v_3^2 = |\vec{v}|^2$$

So we define

$$\|f\|_{L^2} := \left(\langle f,f\rangle\right)^{1/2} = \left(\int f^2\right)^{1/2}.$$

2

This makes L^2 into a metric space:

$$d(f,g) := \|f-g\|_{L^2}.$$

Thus L^2 is a vect. sp. w/ norm and an "inner" product.

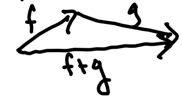
[Why "inner"? Consider $\vec{v},\vec{w}\in\mathbb{R}^n$, say $\vec{v} = \begin{pmatrix}\vdots\end{pmatrix} = n\times 1$, $\vec{w} = \begin{pmatrix}\vdots\end{pmatrix} = n\times 1$. Can almost multiply these:

3

$\vec{v}^t\vec{w} = \vec{v}\cdot\vec{w}$ ←

$\vec{v}\,\vec{w}^t$ ~~= n×n matrix~~. [t is inside.]

All sorts of nice identities and inequalities hold relating inner products and norms:



① $\|f+g\|\le\|f\|+\|g\|$

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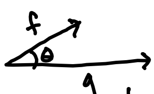
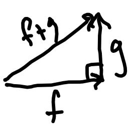


② $|\langle f,g\rangle|\le\|f\|\cdot\|g\|$

"Cauchy-Schwarz inequality"



③ $\|f\|^2+\|g\|^2 = \|f+g\|^2$ when $\langle f,g\rangle = 0$.

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④ $\|f+g\|^2 = \|f\|^2+\|g\|^2 + 2\langle f,g\rangle$

"Law of cosines-ish"

"Polarization property"

"Pf":

$\|f+g\|^2 = \langle f+g, f+g\rangle = \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle =$

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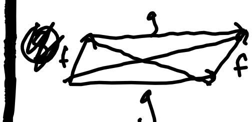
$\|f\|^2 + 2\langle f,g\rangle + \|g\|^2$. ∎



Sum of squares of sides = sum of squares of diags

⑤ "Parallelogram Law":

$$2\|f\|^2 + 2\|g\|^2 = \|f+g\|^2 + \|f-g\|^2$$

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<u>Thm (Riesz-Fischer):</u> L^2 is complete.

Thus, L^2 is nicer that L^1: it has lngth and angles.

Why not come up w/ inner product on L^1?

Using polarization ident,

2

given the L^1 norm we can ~~create~~ define an inner product on L^1!

This doesn't work b/c $\|\cdot\|_{L^1}$ fails parallelogram law, so it <u>can't</u> be induced from an inner product.

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<u>Def$^{\underline{n}}$</u>: Given vect. sp. $V/\mathbb{R}$. An <u>inner product</u> on V is a $f^{\underline{n}}$ $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$

s.t. (1) $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$.

(2) $\langle \alpha \vec{v}, \vec{w} \rangle = \alpha \langle \vec{v}, \vec{w} \rangle$

(3) $\langle \vec{v} + \vec{w}, \vec{z} \rangle = \langle \vec{v}, \vec{z} \rangle + \langle \vec{w}, \vec{z} \rangle$

(4) $\langle \vec{v}, \vec{v} \rangle \geq 0$ w/equality iff $\vec{v} = \vec{0}$.

4

L^1 is a complete normed vect. sp.

"Banach space"

L^2 is a complete ~~normed~~ inner product space

"Hilbert space".

5

Can generalize

$$L^p := \{ f : \int |f|^p < \infty \}$$

w/ $$\|f\|_{L^p} := \left(\int |f|^p \right)^{1/p}$$

$\forall p \geq 1$ this is a norm.

All of these are Banach.

But L^2 is <u>unique</u> Hilbert space.

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