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MATH 402 : MEASURE THEORY

Problem Set 1 – due Monday, September 21st

INSTRUCTIONS:

This assignment must be submitted to Glow before Monday at **6pm**. Late assignments may be submitted by 3pm on Tuesday; however, 5% will be deducted for submissions past Monday 6pm. Assignments will not be accepted after 3pm Tuesday under any circumstances. Please label your file in the format `Lastname-PS1`. If you're having difficulty scanning your work in a way that's legible, please let me or the TA know and we can try to help.

Throughout the problem set, we will follow our idea from class and define a *length* to be any function

$$\ell : \mathcal{P}(\mathbb{R}) \to [0, \infty]$$

that satisfies

(i) $\ell(A \cup B) = \ell(A) + \ell(B)$ for any disjoint sets A, B, and

(ii) $\ell([a, b]) = b - a$ for any real numbers $a \leq b$.

1.1 Suppose ℓ is a length. Here you will deduce some of the properties we came up with in class are consequences of the above definition.

(a) Prove that for arbitrary sets $A, B \subseteq \mathbb{R}$ we have $\ell(A \cup B) \leq \ell(A) + \ell(B)$.

(b) Prove that if $A \subseteq B$ then $\ell(A) \leq \ell(B)$.

(c) Prove that $\ell(\emptyset) = 0$ and $\ell(\mathbb{R}) = \infty$.

1.2 Suppose ℓ is a length, except that rather than satisfying property **(ii)** above it satisfies

$$\textbf{(ii}'\textbf{)}\ \ell\big((a, b)\big) = b - a \text{ for any real numbers } a < b.$$

Prove that ℓ must satisfy **(ii)**.

1.3 (Challenge question!) Given ℓ a length.

(a) Is it true that ℓ must be translation invariant? In other words, is it true that for any $\lambda \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ we have $\ell(\lambda + A) = \ell(A)$?

(b) Is it true that ℓ scales? In other words, is it true that for any $\lambda \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ we have $\ell(\lambda A) = \lambda \ell(A)$?