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MATH 402 : MEASURE THEORY

Problem Set 2 – due Monday, September 28th

INSTRUCTIONS:

This assignment should be submitted to Glow before Monday at **6pm**. Late assignments may be submitted by 3pm on Tuesday; however, 5% will be deducted for submissions past Monday 6pm. Assignments will not be accepted after 3pm Tuesday under any circumstances. Please label your file in the format `Lastname-PS2`. If you're having difficulty scanning your work in a way that's legible, please let me or the TA know and we can try to help.

2.1 Suppose $\{I_1, I_2, \ldots, I_n\}$ is a finite collection of closed intervals that covers the closed interval I, i.e. $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$. Prove that $|I| \leq |I_1| + |I_2| + \cdots + |I_n|$.

2.2 After some playing around, one might decide that the function μ defined by

$$\mu(E) := \lim_{N\to\infty} \frac{1}{N} \# \left(E \cap \frac{1}{N}\mathbb{Z} \right)$$

might be a reasonable notion of length on $\mathbb{R}$. (Here $\frac{1}{N}\mathbb{Z} := \left\{ \frac{k}{N} : k \in \mathbb{Z} \right\}$, and $\#S$ denotes the number of elements in S.) The purpose of this problem is to investigate this.

(a) Prove that $\mu([a, b]) = b - a$ whenever $b \geq a$.

(b) Prove that if A is a finite union of disjoint intervals, then $\mu(A)$ is the sum of the lengths of these intervals.

(c) Construct a set $A \subseteq \mathbb{R}$ for which $\mu(A)$ is undefined. (Note: we consider $\mu(A) = \infty$ well-defined.)

(d) Prove that $\mu(\mathbb{Q} \cap [0, 1]) \neq \mu\left(\sqrt{2} + \mathbb{Q} \cap [0, 1]\right)$, where $\alpha + S$ denotes the translation of the set S by α.

2.3 Textbook questions: Exercises 1, 2, 14, 24, 28 (These are on pages 37–44; Exercises $\neq$ Problems!)