Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 402 : MEASURE THEORY

Problem Set 4 – due Monday, October 26th

INSTRUCTIONS:

You should aim to submit this assignment (via Glow) before Monday at **6pm**. Late assignments may be submitted by **3pm** on Tuesday; however, 5% will be deducted for submissions past Monday 6pm. Assignments will not be accepted after 3pm Tuesday under any circumstances. Please label your file in the format `Lastname-PS4`. If you're having difficulty scanning your work in a way that's legible, please let me or the TA know and we can try to help.

4.1 Recall that a *Vitali set* is any set $X \subseteq \mathbb{R}$ such that

$$\mathbb{R} = \bigsqcup_{q \in \mathbb{Q}} (X + q).$$

Prove that any nonempty open interval $\mathcal{O} \subseteq [0,1]$ contains a Vitali set.

4.2 Write $x \sim y$ iff $x - y \in \mathbb{Q}$, and define

$$[\alpha] := \{x \in [0,1) : x \sim \alpha\}.$$

Let $\Sigma := \{[\alpha] : \alpha \in [0,1)\}$, and let f be any choice function on Σ, i.e. $f : \Sigma \to [0,1)$ such that $f(\sigma) \in \sigma$ for all $\sigma \in \Sigma$. Prove that f is injective.

4.3 In this problem we prove a different version of Banach-Tarski: that a circle can be cut into countably many pieces that, when rearranged, form two copies of the original circle. For notational convenience, rather than working with literal circles we'll work with the interval $I := [0,1)$ under the operation $\oplus$, defined to be addition (mod 1). More precisely, for all $a, b \in I$ we set

$$a \oplus b := \begin{cases} a + b & \text{if } a + b < 1 \\ a + b - 1 & \text{otherwise.} \end{cases}$$

Let Σ and f be as in Problem **4.2**, and set $X := f(\Sigma)$. Finally, for any $q \in \mathbb{Q}$ define $X_q := X \oplus q$.

(a) Prove that $I = \bigsqcup_{q \in \mathbb{Q} \cap I} X_q$.

(b) Find a function $g : \mathbb{Q} \to \mathbb{R}$ such that $I = \bigsqcup_{q \in \mathbb{Q} \cap [0, \frac{1}{2})} (X_q \oplus g(q))$. Justify with a proof.

(c) Find a function $h : \mathbb{Q} \to \mathbb{R}$ such that $I = \bigsqcup_{q \in \mathbb{Q} \cap [\frac{1}{2}, 1)} (X_q \oplus h(q))$. Justify with a proof.

4.4 Textbook exercises[1] 8, 13, 21, 32, 33

[1] These are on pages 37–44; Exercises $\neq$ Problems!