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MATH 402 : MEASURE THEORY

Problem Set 5 – due Monday, November 2nd

INSTRUCTIONS:

You should aim to submit this assignment (via Glow) before Monday at **6pm**. Late assignments may be submitted by **3pm** on Tuesday; however, 5% will be deducted for submissions past Monday 6pm. Assignments will not be accepted after 3pm Tuesday under any circumstances. Please label your file in the format `Lastname-PS5`. If you're having difficulty scanning your work in a way that's legible, please let me or the TA know and we can try to help.

5.0 Read pages 23–34 in Stein-Shakarchi.

5.1 The goal of this problem is to highlight the phenomenon Ben pointed out in class: it's possible for f_n to be a nice sequence of functions that converges to f, but for $\int f_n$ to not converge to $\int f$. Enumerate $\mathbb{Q}$ as $\{q_1, q_2, q_3, \ldots\}$ and consider the sequence of functions

$$f_n := \chi_{\{q_1, q_2, \ldots, q_n\}}.$$

(As usual, χ_A denotes the characteristic function of A.)

(a) Prove that f_n is a monotonically increasing sequence of functions, i.e. that $m \leq n$ implies $f_m(t) \leq f_n(t)$ for all t.

(b) Prove that $\lim\limits_{n\to\infty} f_n = \chi_{\mathbb{Q}}$ pointwise.

(c) Prove that $\displaystyle\int_0^1 f_n = 0$ for any n.

5.2 In most calculus courses the Riemann integral $\int_a^b f$ is presented in terms of left and right Riemann sums: given a positive integer N, set $\Delta_N := \frac{b-a}{N}$ and define

$$\text{Left}(f, N) := \Delta_N \sum_{k=0}^{N-1} f(a + k\Delta_N) \qquad \text{and} \qquad \text{Right}(f, N) := \Delta_N \sum_{k=1}^{N} f(a + k\Delta_N).$$

(Colloquially: divide the interval $[a, b]$ into N identical subintervals and approximate the area under f by a sum of rectangles with height based on either the left endpoint or right endpoint of each subinterval. Draw a picture for yourself if this isn't clear.) If $\lim\limits_{N\to\infty} \text{Left}(f, N) = \lim\limits_{N\to\infty} \text{Right}(f, N)$, then (one is taught) the integral exists and is defined

$$\int_a^b f := \lim_{N\to\infty} \text{Left}(f, N).$$

The goal of this problem is to show that this is not a good definition.

(a) Let $S := \{a + k\Delta_N : k \in \mathbb{Z} \cap [0, N-1]\}$, where a, b, N, Δ_N are defined above. Prove that either $S \subseteq \mathbb{Q}$ or $\#(S \cap \mathbb{Q}) \leq 1$.

(b) Find a function $f : [0,1] \to \mathbb{R}$ and $\alpha \in [0,1]$ such that $\int_a^b f$ exists for every $[a,b] \subseteq [0,1]$ according to the above definition, but

$$\int_0^\alpha f + \int_\alpha^1 f \neq \int_0^1 f.$$

Prove all these assertions.

5.3 Recall that in class we argued (using an implicit LEO) that for any function $f : \mathbb{R} \to \mathbb{R}$ with period 1 we should expect the formula

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e(nx), \tag{$*$}$$

where $a_n := \int_0^1 f(t)e(-nt)\,dt$ and $e(\alpha) := e^{2\pi i\alpha}$. The numbers a_n are called the *Fourier coefficients* of f, and the right hand side of $(*)$ is called the *Fourier series* of f. Although $(*)$ doesn't hold in general (that's the problem with arguments using LEOs!), it turns out it usually does, and a lot of Fourier analysis is concerned with finding conditions under which a function is equal to its Fourier series. A typical classical result in the area is that if f has period 1 and $\int_0^1 |f| < \infty$, then $(*)$ holds for every x at which f is differentiable. How much can one weaken these hypotheses? Quite a lot: one of the biggest breakthroughs of 20th century mathematics was Carleson's theorem (1966) that if $\int_0^1 |f|^2 < \infty$, then $(*)$ holds for almost every x.

The goal of this problem is to see a cool application of Fourier series that will hopefully whet your appetite to learn more about the subject.

(a) Let $F(x) = \{x\}(1 - \{x\})$, where $\{x\}$ denotes the fractional part of x (i.e. $\{x\} = x \pmod 1$, or equivalently, $\{x\} = x - \lfloor x \rfloor$). Prove that $a_0 = \frac{1}{6}$ and $a_n = -\frac{1}{2\pi^2 n^2}$ for all $n \neq 0$, where the a_n denote the Fourier coefficients of F.

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

You may freely use any of the results stated in the introduction to this problem.

(c) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

5.4 Textbook exercises[1] 17, 18, 22

5.5 Textbook problems[2] 1, 4

[1] These are on pages 37–44; Exercises $\neq$ Problems!

[2] These are on pages 46–48; Exercises $\neq$ Problems!