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MATH 402 : MEASURE THEORY

Problem Set 6 – due Monday, November 9th

INSTRUCTIONS:

You should aim to submit this assignment (via Glow) before Monday at **6pm**. Late assignments may be submitted by **3pm** on Tuesday; however, 5% will be deducted for submissions past Monday 6pm. Assignments will not be accepted after 3pm Tuesday under any circumstances. Please label your file in the format `Lastname-PS6`. If you're having difficulty scanning your work in a way that's legible, please let me or the TA know and we can try to help.

6.0 Think about how to fix Cauchy's proof. You don't have to submit this with the assignment!

6.1 Abel famously constructed the following counterexample to Cauchy's theorem:

$$f(x) = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$

The goal of this problem is to show that f is a counterexample, and also, that it's not nearly as bizarre a function as it first appears.

(a) Prove that f isn't continuous everywhere on $\mathbb{R}$.

(b) Write $f'(x)$ in closed form, i.e. without sigma notation or ellipses. Don't fret about convergence or LEOs.

(c) Write $f(x)$ in closed form.

6.2 On a previous problem set you proved that it's possible for the continuous image of a measurable set to be non-measurable. The goal of this problem is show that it's possible for the continuous preimage of a measurable set to be non-measurable, as well. Let $F : \mathcal{C} \to \mathbb{R}$ be the Cantor-Lebesgue function, and recall that F can be extended to a continuous function $[0, 1] \to \mathbb{R}$ as in exercise 2(d). Now we form a new function $G : \mathbb{R} \to \mathbb{R}$ defined by

$$G(x) := \begin{cases} x & \text{if } x < 0 \\ x + 1 & \text{if } x > 1 \\ x + F(x) & \text{if } x \in [0, 1]. \end{cases}$$

(a) Prove that G is a continuous bijection, and that G^{-1} is continuous. (In fancy language, this says G is a *homeomorphism*.)

(b) Prove that $m\Big(G(\mathcal{C})\Big) = 1$, where $\mathcal{C}$ denotes the Cantor set.

(c) Prove that there exists a non-measurable set $\mathcal{N}$ such that $G^{-1}(\mathcal{N})$ is measurable.

(d) Prove that there exists a measurable function $H : \mathbb{R} \to \mathbb{R}$ and a measurable set A such that $H^{-1}(A)$ is non-measurable.

6.3 Suppose $h : \mathbb{R} \to \mathbb{R}$ is an arbitrary measurable function and B is a Borel set. Prove that $h^{-1}(B)$ must be measurable.

6.4 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. In class we saw that f must be Lebesgue-to-Borel measurable. Must f be Lebesgue-to-Lebesgue measurable?

6.5 Textbook exercises[1] 34, 35

[1] These are on pages 37–44; Exercises $\neq$ Problems!