Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 402 : MEASURE THEORY

Problem Set 7 – due Monday, November 16th

INSTRUCTIONS:

You should aim to submit this assignment (via Glow) before Monday at **6pm**. Late assignments may be submitted by **3pm** on Tuesday; however, 5% will be deducted for submissions past Monday 6pm. Assignments will not be accepted after 3pm Tuesday under any circumstances. Please label your file in the format `Lastname-PS7`. If you're having difficulty scanning your work in a way that's legible, please let me or the TA know and we can try to help.

7.1 Let $I := [0,1]$. Lusin's theorem implies the existence of $F \subseteq I$ such that $m(I \setminus F) < \epsilon$ and $\chi_{\mathbb{Q}\cap I}\Big|_F$ is continuous. Give an explicit example of such an F.

7.2 Carefully write down the proof of the following: if $f_n : E \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to f, then f is continuous.

7.3 Consider the function $\sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$\sigma(x) := \frac{1}{2} - \left| \{x\} - \frac{1}{2} \right|,$$

where $\{x\}$ denotes the fractional part of x (i.e. $\{x\} = x \pmod 1 = x - \lfloor x \rfloor$). Out of this we now build a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined

$$f_n(x) := \frac{1}{2^n}\sigma(2^n x).$$

(a) Give a simple description of the function $f(x) := \lim_{n\to\infty} f_n(x)$.

(b) Does $f_n \to f$ uniformly? Prove your assertion.

7.4 Suppose $f_n : [0,1] \to \mathbb{R}$ are continuous functions such that $f_n \nearrow f$ (i.e. f_n converges to f pointwise and $f_n(x) \le f_{n+1}(x)$ for every n and x). Prove that f is continuous iff $f_n \to f$ uniformly. [*Hint. Consider the sets* $A_n := \{x : f(x) - f_n(x) < \epsilon\}$.]

7.5 Show by example that Egorov's theorem cannot be improved to an 'almost everywhere' result. More precisely, construct a sequence of measurable functions $f_n : [0,1] \to \mathbb{R}$ that converge almost everywhere to some function $f : [0,1] \to \mathbb{R}$ pointwise, but that don't converge uniformly to f on any closed set of measure 1.

7.6 Suppose $F \subseteq \mathbb{R}$ is closed, and that $f : F \to \mathbb{R}$ is continuous on F. Prove that there exists a function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{f}$ is continuous on $\mathbb{R}$ and $\widetilde{f}\Big|_F = f$.

7.7 Extend Lusin's theorem to the case that $m(E)$ is infinite. [*Hint. Use the finite version of Lusin's theorem.*]