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# MATH 402 : MEASURE THEORY

## Problem Set 8 – due Tuesday, December 1st

### INSTRUCTIONS:

This assignment should be submitted by **3pm on Tuesday**. (Note that there will be no late penalty this week for Tuesday submission. Please do not submit the assignment past the deadline, however.) Please label your file in the format `Lastname-PS8`. If you're having difficulty scanning your work in a way that's legible, please let me or the TA know and we can try to help.

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**8.0** Read pages 49–58 in Stein-Shakarchi.

**8.1** In lecture 12, we gave a visual description of Lebesgue integration for any non-negative measurable function and derived a formula in terms of a Riemann integral. Evaluate the following integrals using Lebesgue's approach of horizontal slicing. No need for rigor; this problem is about cementing the concept.

(a) $\displaystyle\int_{[-1,1]} x^{-2/3}$

(b) $\displaystyle\int_{[0,1]} f$, where $f(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \\ 1/n & \text{if } x = \frac{m}{n} \text{ in reduced form.} \end{cases}$

**8.2** Now for a rigorous computation. Let $F : [0,1] \to [0,1]$ denote the Cantor-Lebesgue function.

(a) Give an explicit sequence of simple functions $\psi_n$ such that $\psi_n \to F$.

(b) Compute $\displaystyle\int_{[0,1]} \psi_n$.

(c) Compute $\displaystyle\int_{[0,1]} F$.

**8.3** The goal of this problem is to give a short (non-Egorov-based) proof of Lusin's theorem, discovered by Loeb and Talvila in 2004. Throughout, assume $f : E \to \mathbb{R}$ is measurable and that $m(E) < \infty$.

(a) Prove that the set of all open intervals with rational endpoints is countable. Let $\mathcal{O}_n$ be an enumeration of these.

(b) For each $n$, prove that there exist compact sets $A_n \subseteq f^{-1}(\mathcal{O}_n)$ and $B_n \subseteq E \setminus f^{-1}(\mathcal{O}_n)$ such that

$$m(E \setminus (A_n \cup B_n)) < \frac{\epsilon}{2^n}.$$

(c) Let $C := \bigcap_{n=1}^{\infty} (A_n \cup B_n)$. Show that $m(E \setminus C) < \epsilon$.

(d) Prove that $C$ is compact, and that $f\Big|_C$ is continuous.

**8.4** Recall that a measurable $f : \mathbb{R}^d \to \mathbb{R}$ is *simple* iff $f$ is a finite linear combination of characteristic functions of disjoint measurable sets. In class we asserted a couple equivalent definitions. The purpose of this exercise is to prove the equivalence.

(a) Suppose $f = \sum_{n \le N} \alpha_n \chi_{E_n}$, where $\alpha_n \in \mathbb{R}$ and the $E_n$ are measurable (but not necessarily disjoint). Prove that $f$ takes finitely many values. Can you figure out an upper bound on how many values it can take?

(b) Prove that if a measurable $f : \mathbb{R}^d \to \mathbb{R}$ takes finitely many values, it must be simple.

**8.5** Suppose $f : [a, b] \to \mathbb{R}$ is continuous almost everywhere. Prove that $f$ must be measurable. (By Lebesgue's criterion, this gives a quick proof that any Riemann integrable function is measurable.)

**8.6** Does the Bounded Convergence Theorem hold for the Riemann integral? Justify your answer with a proof or a counterexample.

**8.7** Suppose $\{f_n\}$ is a sequence of measurable functions. Consider the assertion

$$\int f_n \to 0 \qquad \Longrightarrow \qquad f_n \to 0 \text{ almost everywhere.} \tag{$*$}$$

Is $(*)$ true? Can you formulate some hypotheses on $f_n$ that allow you to prove that $(*)$ holds?