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MATH 402 : MEASURE THEORY

Problem Set 9 – due Monday, December 7th

INSTRUCTIONS:

You should aim to submit this assignment (via Glow) before Monday at **6pm**. Late assignments may be submitted by **3pm** on Tuesday; however, 5% will be deducted for submissions past Monday 6pm. Assignments will not be accepted after 3pm Tuesday under any circumstances. Please label your file in the format `Lastname-PS9`. If you're having difficulty scanning your work in a way that's legible, please let me or the TA know and we can try to help.

9.0 Read pages 58–64 in Stein-Shakarchi.

9.1 Write down the proof of Fatou's Lemma stated in class (it's a bit different from the book's version). This exercise is intended to give you a chance to check whether you understood the proof from class!

9.2 Given a sequence of functions $f_n : \mathbb{Z}_{>0} \to [0, \infty]$, prove that

$$\sum_{k=1}^{\infty} \liminf_{n\to\infty} f_n(k) \leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} f_n(k)$$

9.3 Here we explore Ben's question from class about a variation on Fatou's lemma. Throughout, assume we're given a sequence of measurable functions $f_n : \mathbb{R}^d \to [0, \infty]$.

(a) Suppose there exists a measurable function g such that $0 \leq f_n \leq g$ for all n and $\int g < \infty$. Prove that

$$\limsup_{n\to\infty} \int f_n \leq \int \limsup_{n\to\infty} f_n.$$

(b) Is the existence of g necessary?

9.4 Suppose $m(E) = 0$ and $f(x) = \infty$ for all $x \in E$. Prove that $\int_E f = 0$.

9.5 Given $\alpha \in \mathbb{R}$, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^\alpha$ on $(0, 1]$ and $f(x) = 0$ elsewhere. Compute $\int_{[0,1]} f$.

9.6 Given a sequence (a_n) of non-negative real numbers. Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) := a_{\lfloor x \rfloor}$. Prove that

$$\int_{[1,\infty)} f = \sum_{n=1}^{\infty} a_n.$$

9.7 Given a measurable $f : \mathbb{R} \to [0, \infty]$. Prove that

$$\lim_{n\to\infty} \int_{[-n,n]} f = \int f.$$

9.8 Textbook, pages 89–95: Exercises 6(a), 9