UNIVERSITY OF TORONTO SCARBOROUGH

MATA31H3 : Calculus for Mathematical Sciences I

MIDTERM EXAMINATION # 1

October 3, 2012

Duration – 2 hours
Aids: none

NAME (PRINT): ______________________________
Last/Surname First/Given Name (and nickname)

STUDENT NO: KEY

TUTORIAL: ______________________________
Tutorial section # Name of TA

Qn. #	Value	Score
COVER PAGE	5	
1	20	
2	20	
3	15	
4	15	
5	10	
6	15	
Total	100	

TOTAL: ____________

Please read the following statement and sign below:

I understand that any breach of academic integrity is a violation of The Code of Behaviour on Academic Matters. By signing below, I pledge to abide by the Code.

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(1) (20 points) Use induction to prove that

$$\sum_{k=1}^{n} 4^k = \frac{4}{3}(4^n - 1) \qquad \forall n \in \mathbb{N}.$$

[*Recall that* $\sum_{k=1}^{n} 4^k$ *is a fancy way of writing* $4^1 + 4^2 + \cdots + 4^n$.]

Proof: By induction.

Let

$$\mathcal{A} = \left\{ n \in \mathbb{N} : \sum_{k=1}^{n} 4^k = \frac{4}{3}(4^n - 1) \right\}.$$

$1 \in \mathcal{A}$, since $4 = \frac{4}{3}(4-1)$.

Now, suppose $n \in \mathcal{A}$. Then from the definition of $\mathcal{A}$,

$$\sum_{k=1}^{n} 4^k = \frac{4}{3}(4^n - 1).$$

Adding 4^{n+1} to both sides yields

$$\begin{aligned}
\sum_{k=1}^{n+1} 4^k &= \frac{4}{3}(4^n - 1) + 4^{n+1} \\
&= \frac{4^{n+1} - 4}{3} + \frac{3 \cdot 4^{n+1}}{3} \\
&= \frac{4 \cdot 4^{n+1} - 4}{3} \\
&= \frac{4^{n+2} - 4}{3} \\
&= \frac{4}{3}(4^{n+1} - 1).
\end{aligned}$$

Thus, we see that whenever $n \in \mathcal{A}$, we must have $n + 1 \in \mathcal{A}$ as well. By induction, we conclude that $\mathcal{A} = \mathbb{N}$, which is precisely what is claimed. $\square$

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(2) (20 points) Prove that $\sqrt{5} \notin \mathbb{Q}$.

There are (at least!) two different approaches:

Proof 1: Suppose $\sqrt{5} \in \mathbb{Q}$. Then $\exists a \in \mathbb{Z}$ and $\exists b \in \mathbb{N}$ such that $\frac{a}{b}$ is fully reduced, and

$$\sqrt{5} = \frac{a}{b}.$$

Squaring both sides and simplifying shows that

(†) $$a^2 = 5b^2.$$

In particular, a^2 is a multiple of 5. It follows (see Lemma below) that a is a multiple of 5. So, we can write $a = 5k$ for some $k \in \mathbb{Z}$. Plugging this back into (†) and simplifying yields

$$b^2 = 5k^2.$$

Exactly as above, this implies that b is a multiple of 5. Thus, both a and b must be multiples of 5. But this contradicts our assumption that $\frac{a}{b}$ is fully reduced! $\square$

Lemma: Suppose $a \in \mathbb{Z}$, and a^2 is a multiple of 5. Then a is also a multiple of 5.

Proof. Let k be the smallest integer such that $5k > a$. (Such an integer exists by the Well-Ordering Property of $\mathbb{N}$.) This implies that $5(k-1) \leq a$, since otherwise, k wouldn't be minimal. So, we have

$$5k - 5 \leq a < 5k.$$

This means that

$$a = 5k - r,$$

where $r = 1, 2, 3, 4,$ or 5. Moving things around and squaring both sides, we see that

$$r^2 = a^2 - 10kr + 25k^2.$$

Note that the entire right hand side is a multiple of 5, since a^2 is a multiple of 5 by hypothesis, and $10kr$ and $25k^2$ are clearly multiples of 5. We thus conclude that r^2 must be a multiple of 5. There are only five possible values of r, and it's easy to check that the only one of these for which r^2 is a multiple of 5 is $r = 5$. Thus, we conclude that $a = 5k - 5$, which is a multiple of 5 as claimed. $\square$

Proof 2: Let $\mathcal{A} = \{n \in \mathbb{N} : n\sqrt{5} \in \mathbb{Z}\}$. If $\mathcal{A} = \emptyset$, then we're done.

Suppose instead that $\mathcal{A} \neq \emptyset$. By the Well-Ordering Property of $\mathbb{N}$, $\mathcal{A}$ must have some least element; call it a. I now claim that

(*) $$a(\sqrt{5} - 2) \in \mathcal{A}.$$

Since $a(\sqrt{5} - 2) < a$, this would contradict the minimality of a.

So, why is (*) true? First, note that $a(\sqrt{5} - 2) \in \mathbb{Z}$, since $a\sqrt{5} \in \mathbb{Z}$ and $2a \in \mathbb{Z}$. Next, since $a(\sqrt{5} - 2) > 0$, we deduce that it is actually a natural number. Finally, we have

$$a(\sqrt{5} - 2) \cdot \sqrt{5} = 5a - 2a\sqrt{5} \in \mathbb{Z}.$$

Thus, (*) is true. This contradicts the minimality of a; it follows that $\mathcal{A}$ must be empty after all. $\square$

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(3) (15 points) Suppose $a \in \mathbb{Q}$, $\beta \notin \mathbb{Q}$, and $a \neq 0$. Prove that $a\beta \notin \mathbb{Q}$.

We start with a lemma.

Lemma: If $x, y \in \mathbb{Q}$ and $y \neq 0$, then $x/y \in \mathbb{Q}$.

Proof. We can write $x = m/n$ and $y = k/\ell$, where $m, n, k, \ell \in \mathbb{Z}$ and n, k, and ℓ are all nonzero. Then

$$x/y = \frac{m\ell}{kn}$$

which is clearly rational since $m\ell \in \mathbb{Z}$, $kn \in \mathbb{Z}$, and $kn \neq 0$. $\square$

We can now prove the claim. Let $x = a\beta$. Then

$$\beta = x/a$$

(since $a \neq 0$). By the lemma, if x were rational, then x/a would be as well. But this contradicts our assumption that $\beta \notin \mathbb{Q}$. Thus, x cannot be rational. QED

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(4) (15 points) Let

$$\mathcal{A} = \{n \in \mathbb{N} : 2n - 4 \text{ is a multiple of } 6\}$$

and

$$\mathcal{B} = \{3k - 1 : k \in \mathbb{N}\}.$$

Prove that $\mathcal{A} = \mathcal{B}$. [*Note: 0 is a multiple of 6.*]

Proof. We accomplish this in two steps: we first show that $\mathcal{A} \subseteq \mathcal{B}$, and then we show that $\mathcal{B} \subseteq \mathcal{A}$.

$\underline{\mathcal{A} \subseteq \mathcal{B}:}$

Pick any $n \in \mathcal{A}$. Then $2n-4 = 6\ell$ for some $\ell \in \mathbb{Z}$. (Moreover, since $n \geq 1$, we must have $6\ell \geq -2$; since $\ell \in \mathbb{Z}$, it follows that $\ell \geq 0$.) Solving for n, we find: $n = 3\ell + 2 = 3(\ell + 1) - 1$. Since $\ell \geq 0$ is an integer, we conclude that $\ell + 1 \in \mathbb{N}$. Thus, $n \in \mathcal{B}$.

$\underline{\mathcal{B} \subseteq \mathcal{A}:}$

Pick any $m \in \mathcal{B}$. Then $m = 3k - 1$ for some $k \in \mathbb{N}$. This immediately implies that $m \in \mathbb{N}$ (since $k \geq 1 \implies 3k - 1 \geq 2$). Moreover,

$$2m - 4 = 2(3k - 1) - 4 = 6k - 6 = 6(k - 1),$$

which is clearly a multiple of 6. It follows that $m \in \mathcal{A}$.

Since $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$, we conclude that $\mathcal{A} = \mathcal{B}$. $\square$

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(5) (10 points) What's wrong with the following induction proof? (Note: the claim is false, but that's not what's wrong *with the proof*.)

Claim: $2n - 1 \leq \frac{n^2+1}{2}$ for all $n \in \mathbb{N}$.

'Proof': By induction. Let

$$\mathcal{A} = \left\{ n \in \mathbb{N} : 2n - 1 \leq \frac{n^2+1}{2} \right\}$$

We have $1 \in \mathcal{A}$, since $2(1) - 1 = 1 = \frac{1^2+1}{2}$.

Now, suppose $n \in \mathcal{A}$, so that $2n - 1 \leq \frac{n^2+1}{2}$. Then

$$\begin{aligned}
\frac{(n+1)^2+1}{2} &= \frac{n^2+2n+1+1}{2} \\
&= \frac{n^2+1}{2} + \frac{2n+1}{2} \\
&\geq 2n - 1 + n + \frac{1}{2} \\
&= 3n - \frac{1}{2} \\
&\geq 2n - 1.
\end{aligned}$$

It follows that $n + 1 \in \mathcal{A}$.

By induction, we conclude that $\mathcal{A} = \mathbb{N}$; this is precisely the claim. 'QED'

It is shown in the proof that if $n \in \mathcal{A}$, then

$$2n - 1 \leq \frac{(n+1)^2+1}{2}.$$

However, this does not imply that $n + 1 \in \mathcal{A}$; for that, we would have had to show that

$$2(n+1) - 1 \leq \frac{(n+1)^2+1}{2}.$$

[Incidentally, the claim is false for $n = 2$, but is true for every other natural number n.]

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(6) (15 points) Prove that $\sqrt{2} + \sqrt{6} \notin \mathbb{Q}$.

Many people tried to prove this by claiming that the sum of two irrationals must be irrational. This is not a viable approach, because the sum of two irrational numbers *might be rational*. For example, $3 - \sqrt{2}$ and $\sqrt{2}$ are both irrational, but their sum is rational.

There are several ways to prove that $\sqrt{2} + \sqrt{6} \notin \mathbb{Q}$. Here's one slick method.

Proof. Let $\alpha = \sqrt{2} + \sqrt{6}$, and set $\beta = \sqrt{6} - \sqrt{2}$. Then $\alpha\beta = 4 \in \mathbb{Q}$. By problem (3) on this exam, this means that either both α and β are rational, or both are irrational.

Now, note that $\alpha - \beta = 2\sqrt{2}$, which is irrational (again by problem (3) of this exam). This is clearly impossible if both α and β were rational. Thus, both must be irrational. In particular, $\alpha = \sqrt{2} + \sqrt{6} \notin \mathbb{Q}$. $\square$

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Total Marks = 100 points