LECTURE SUMMARY

Today, we finished our discussion of induction by proving that the Well-Ordering Property of $\mathbb{N}$, Induction, and Strong Induction are all equivalent to one another. More precisely, we proved three theorems.

Theorem 1. *The Well-Ordering Property of* $\mathbb{N}$ *implies Induction.*

Proof. This one is in Bartle and Sherbert (Theorem 1.2.2). $\square$

Theorem 2. *Induction implies Strong Induction.*

Proof. Given a set $\mathcal{A} \subseteq \mathbb{N}$ such that $1 \in \mathcal{A}$, and that whenever $\{1, 2, \ldots, n\} \subseteq \mathcal{A}$, $n + 1$ must belong to $\mathcal{A}$ as well. We wish to prove that $\mathcal{A}$ must be all of $\mathbb{N}$.

Let $\mathcal{B} = \big\{n \in \mathbb{N} : \{1, 2, \cdots, n\} \subseteq \mathcal{A}\big\}$. I now claim that $\mathcal{B} = \mathbb{N}$. To prove this, I use induction. First, we clearly have $1 \in \mathcal{B}$. Next, suppose $n \in \mathcal{B}$. Then by definition, $\{1, 2, \ldots, n\} \subseteq \mathcal{A}$. By our hypothesis on $\mathcal{A}$, this implies that $n + 1 \in \mathcal{A}$. Thus, we see that $\{1, 2, \ldots, n, n + 1\} \subseteq \mathcal{A}$. It follows that $n + 1 \in \mathcal{B}$. In short,

$$n \in \mathcal{B} \Longrightarrow n + 1 \in \mathcal{B}.$$

By induction, $\mathcal{B} = \mathbb{N}$. This in turn implies that $\mathcal{A} = \mathbb{N}$, since for any $k \in \mathbb{N}$, we have $k \in \mathcal{B}$, whence $k \in \mathcal{A}$. $\square$

Theorem 3. *Strong Induction implies the Well-Ordering Property of* $\mathbb{N}$.

Proof. Suppose $\mathcal{A} \subseteq \mathbb{N}$ has no least element, and set

$$\overline{\mathcal{A}} = \mathbb{N} \backslash \mathcal{A}.$$

I claim that $\overline{\mathcal{A}} = \mathbb{N}$, and prove this by strong induction. First, note that $1 \in \overline{\mathcal{A}}$; otherwise, it would be the least element of $\mathcal{A}$. Next, suppose $\{1, 2, \ldots, n\} \subseteq \overline{\mathcal{A}}$. Then $n + 1 \in \overline{\mathcal{A}}$ as well, since otherwise, $n + 1$ would be the least element of $\mathcal{A}$. Thus, by strong induction, $\overline{A} = \mathbb{N}$. It follows that $\mathcal{A} = \emptyset$. In other words, every subset of $\mathbb{N}$ with no least element must be empty; this is precisely the Well-Ordering Property of $\mathbb{N}$. $\square$

Note that the above theorems don't prove that any of WOP, induction, or strong induction are true. What they prove is that either all three statments are true, or all three are false.

Next, we turned to the properties of $\mathbb{R}$. We stated four properties of addition (A1–A4), four properties of multiplication (M1–M4), and one property connecting the two (D). These are all in section 2.1 of the textbook, although I stated A2 (and M2) in a slightly different way: for all $a, b, c \in \mathbb{R}$,

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the symbol $a + b + c$ is unambiguous. For example, $4 + 2 + 2$ is 8. By contrast, $4 \div 2 \div 2$ might be either 1 or 4, depending on where you put parentheses.

We finished class by proving (using only these properties) that $a \cdot 0 = 0$ for all $a \in \mathbb{R}$, and also that 0 is unique. More precisely:

Theorem 4. *Suppose that for all $a \in \mathbb{R}$, we have $a + 0 = a$ and $a + 0' = a$. Then $0 = 0'$.*

Proof. Consider $0 + 0'$. On one hand, it's equal to 0. On the other it equals $0'$. Thus, $0 = 0'$. $\square$