TWO PROOFS OF THE IRRATIONALITY OF $\sqrt{2}$

Theorem 1. $\sqrt{2} \notin \mathbb{Q}$

First proof. Suppose $\sqrt{2}$ were rational. Then we could write

$$\sqrt{2} = \frac{a}{b}$$

where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $\frac{a}{b}$ is fully reduced.

After some easy algebraic manipulations, we see that

$$a^2 = 2b^2. \tag{*}$$

In particular, a^2 must be even. It follows (why?) that a is even, so we can write $a = 2c$, where $c \in \mathbb{Z}$. Plugging this back into the equation (*) and simplifying, we see that $b^2 = 2c^2$. As before, this implies that b^2 is even, so b must be even.

We've therefore shown that if $\sqrt{2}$ could be written as a (reduced) fraction $\frac{a}{b}$, then both a and b are even. But then $\frac{a}{b}$ isn't reduced! Since every step of our argument was logical, the only thing which can be wrong is our initial assumption that $\sqrt{2}$ is rational. This concludes the proof. □

Second proof. Consider the set

$$A = \{n \in \mathbb{N} : n\sqrt{2} \in \mathbb{Z}\}.$$

If this set is empty, then we're done with the proof. (Why?)

Suppose instead that A is not empty. Then it has some smallest element, say, a. I now claim:

(i) $a(\sqrt{2} - 1) \in A$; and
(ii) $a(\sqrt{2} - 1) < a$.

(Why are these true?) This contradicts the minimality of a. It follows that A has no smallest element, which is obviously impossible unless A is empty. □

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