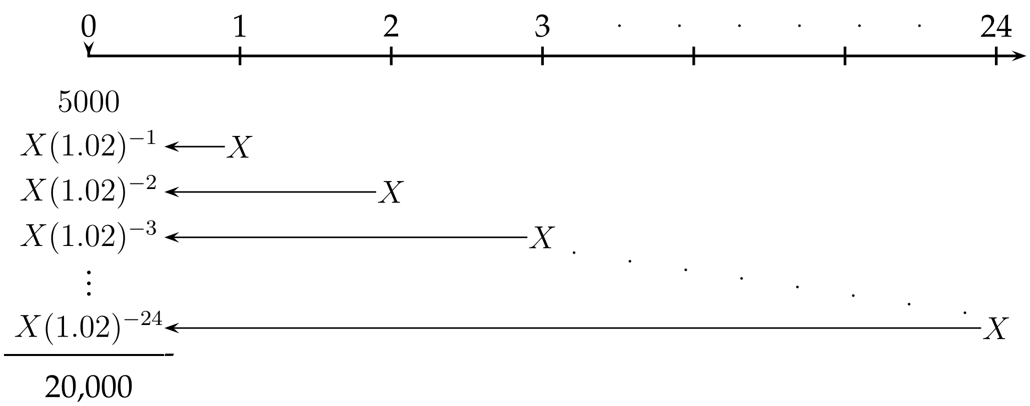
Midterm Solutions

1. First, from the definition, it is clear that $f(x)$ is continuous for $x \neq 5$. Thus, the only possible discontinuity is at $x = 5$. Note that $\lim_{x \to 5^-} f(x) = 15 + 2k^2$ and $\lim_{x \to 5^+} f(x) = 27 + 2k$. Therefore, for $f(x)$ to be continuous at 5, we would need

$$15 + 2k^2 = 27 + 2k.$$

This is a quadratic equation; solving it, one finds that either $k = 3$ or $k = -2$.

2. (a) Consider the following time diagram:



As can be seen from the picture, the sum of the items in the first column must equal 20000. Summing the geometric series, we see that $X \times \dfrac{1 - (1.01)^{-24}}{0.01} = 15000$. Solving for X gives $X = \dfrac{150}{1 - (1.01)^{-24}} \approx 706.102$. Therefore, the first installment is approximately \$706.10.

(b) The finance charge is the total interest paid. Since the total amount paid is $5000 + 24X$, and the cost of the car is \$20000, the amount paid which went towards interest is $5000 + 24X - 20000$. Thus, the finance charge is approximately \$1946.45.

3. Let n be the number of years required. Then

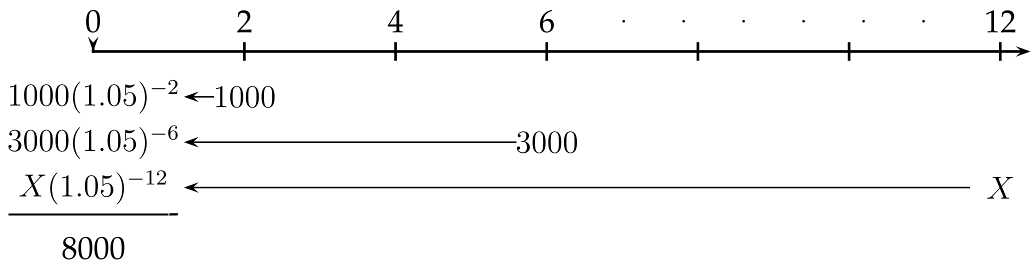
$$P\left(1 + \frac{0.08}{2}\right)^{2n} = 1.5P.$$

Taking logarithms of both sides, we find that

$$n = \frac{\log 1.04}{2 \log 1.5} \approx 5.169$$

However, compounding only occurs every half-year, so P won't have increased by 50% until 5.5 years have gone by. Converting this to months, we obtain our answer: 66 months.

4. Consider the following time diagram:



(Alternatively, one could draw the diagram with future values of the 1000, 3000, and 8000.)

From the picture, we see that the sum of the three PVs must equal \$8000. Solving this for X yields $X = 8000(1.05)^{12} - 1000(1.05)^{10} - 3000(1.05)^{6} \approx 8717.669$. Therefore, the final payment would be approximately \$8717.67.

5. The function can be bad when the denominator is zero, or when the fraction underneath the square-root is negative. When does this happen? We make the following observations:

- when $x > 2$, both $2^x - 2$ and $x - 2$ are positive (so the square-root of their quotient exists;
- when $x = 2$, $2^x - 2 = 2$ and $x - 2 = 0$;
- for $1 < x < 2$, $2^x - 2 > 0$ while $x - 2 < 0$, making the fraction under the square-root negative (so $g(x)$ does not exists for such x).
- for $x \leq 1$, $2^x - 2 \leq 0$ and $x - 2 < 0$, making the fraction under the square-root positive and therefore implying that $g(x)$ exists in this range.

Inspecting these observations, we find the following answers:

(i) **d**: $\sqrt{\frac{1}{2}}$

(ii) **h**: none of the above (DNE)

(iii) **h**: none of the above (DNE)

(iv) **f**: ∞

(v) **a**: 0

(vi) **f**: ∞

6. Let $A(t)$ denote the amount of of Carbon-14 present, t years after the tool was made, and let X be the age of the tool. Since it is an exponential function,

$$A(t) = ab^t$$

for some constants a and b. Plugging in $t = 0$ gives $a = A(0)$. Because the half-life is 5730 years, we know that

$$A(5730) = \frac{1}{2}A(0).$$

On the other hand, $A(5730) = ab^{5730} = A(0)b^{5730}$. This gives

$$\frac{1}{2}A(0) = A(0)b^{5730}.$$

Solving for b gives

$$b = \left(\frac{1}{2}\right)^{1/5730}.$$

We also know that the amount of C-14 is 20% of the original amount, i.e. $A(X) = 0.2A(0)$. But from above,

$$A(X) = ab^X = A(0)\left(\frac{1}{2}\right)^{X/5730}$$

Setting this equal to $0.2A(0)$ and solving gives $X = \frac{5730 \log 0.2}{\log 0.5} \approx 13304.648$ years.