MATA32 – Winter 2010
Examples of u-substitution

EXAMPLE I: Evaluate $\displaystyle\int_1^2 \frac{x^2}{1+5x^3}\,dx$.

The first thing one should try is to use the definition of the integral, i.e. using geometry. In this case this is hopeless – the only shapes for which we can find areas are either made of lines or pieces of circles, whereas this function is some crazy thing.

The second thing one should try is to use the fundamental theorem of calculus. In this case, it's really not easy to do so, since it's not at all obvious what an antiderivative of the function $\dfrac{x^2}{1+5x^3}$ is.

Since both of the above approaches fail, we resort to a third: u-substitution. The idea is this: right now, the integral is all in terms of x; perhaps the expression becomes easier to integrate if we cleverly rewrite the integral (in terms of a new variable, u).

Let $u = 1+5x^3$. Then $\frac{du}{dx} = 15x^2$, so (as in lecture) we write $du = 15x^2\,dx$. In our integral we have just $x^2\,dx$, so we write this in terms of du:

$$x^2\,dx = \frac{1}{15}\,du.$$

Putting this all together, we see that the inside of the integral (called the *integrand*) can now be rewritten in terms of u in a pretty simple way:

$$\frac{x^2}{1+5x^3}\,dx = \frac{1}{u} \times \frac{1}{15}\,du$$

Finally, in the original integral, x started at 1 and ended at 2. Since we're writing the new integral all in terms of u, we have to figure out the new bounds of integration. This isn't too hard: recall that $u = 1+5x^3$, so when $x = 1$, we have $u = 6$, and when $x = 2$, $u = 41$. Thus, we have:

$$\begin{aligned}
\int_1^2 \frac{x^2}{1+5x^3}\,dx &= \frac{1}{15}\int_6^{41} \frac{1}{u}\,du \\
&= \frac{1}{15}\ln u\Big|_6^{41} \\
&= \frac{1}{15}\ln 41 - \frac{1}{15}\ln 6
\end{aligned}$$

EXAMPLE II: Evaluate $\displaystyle\int_2^3 \frac{5t}{(3-2t^2)^2}\,dt$.

As before, neither geometry nor the fundamental theorem directly apply, so we go to u-substitution. Let $u = 3 - 2t^2$. Then $du = -4t\,dt$, and we deduce that

$$\frac{5t}{(3-2t^2)^2}\,dt = 5 \times \frac{1}{u^2} \times \left(-\frac{1}{4}\right)\,du$$

Finally, we change the bounds of integration: when $x = 2$, $u = -5$, and when $x = 3$, $u = -15$, so:

$$\begin{aligned}\int_2^3 \frac{5t}{(3-2t^2)^2}\,dt &= -\frac{5}{4}\int_{-5}^{-15} \frac{1}{u^2}\,du \\ &= -\frac{5}{4} \times \left(-\frac{1}{u}\right)\Big|_{-5}^{-15} \\ &= \frac{5}{4}\left(-\frac{1}{15} + \frac{1}{5}\right) \\ &= \frac{1}{6}\end{aligned}$$

Practice problems:

(1) $\displaystyle\int_1^4 4xe^{2x^2}\,dx$

(2) $\displaystyle\int_0^1 \frac{e^{2x}}{1+e^{2x}}\,dx$

(3) $\displaystyle\int_2^3 \frac{e^{1/x}}{x^2}\,dx$

Ans:
(1) $e^{32} - e^2$
(2) $\frac{1}{2}\ln(1+e^2) - \frac{1}{2}\ln 2$
(3) $e^{1/2} - e^{1/3}$