THE SECOND DERIVATIVE TEST

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Suppose a function $F(x)$ satisfies $F'(a) = 0$. In many situations, it is easy to determine whether F has a maximum, minimum, or neither at a by considering the behaviour of the derivative F' slightly to the left and right of a. For example, we saw in lecture that the function $F(x) = \frac{1}{1+x^2}$ must have a maximum at 0, since $F'(x) > 0$ for all $x < 0$ (so F is increasing to the left of 0), and $F'(x) < 0$ for all $x > 0$ (so F is decreasing to the right of 0).

This approach always works in principle, but is occasionally difficult in practice, as we saw with the example $g(x) = x^5+10x^3-80x+2$. We determined easily enough that $g'(x) = 0$ iff $x = \pm\sqrt{2}$. The next natural question is, what is the behaviour of g at these points? In principle, one can do the same procedure as above: determine the behaviour of $g'(x)$ slightly to the right and left of $\pm\sqrt{2}$, and go from there. This is not so easy to do without a calculator! However, there's a trick. Instead of considering the first derivative near $\sqrt{2}$, for example, we considered the second derivative *at* $\sqrt{2}$. We have $g''(x) = 20x(x^2 + 3)$, whence $g''(\sqrt{2}) > 0$. This tells us that $g'(x)$ is increasing in a neighbourhood of $\sqrt{2}$. Since $g'(\sqrt{2}) = 0$, we deduce that $g'(x) < 0$ for x slightly less than $\sqrt{2}$, and $g'(x) > 0$ for x slightly larger than $\sqrt{2}$. This in turn implies that g is decreasing to the left of $\sqrt{2}$, and increasing to the right of $\sqrt{2}$. So, g must have a minimum at $\sqrt{2}$!

More generally, we have the following theorem:

Theorem (Second Derivative Test)**.** *Suppose $g'(a) = 0$ and $g''(a) > 0$. Then g has a local minimum at a. Similarly, if $g'(a) = 0$ and $g''(a) < 0$, then g has a local maximum at a.*

Here's a not entirely rigorous proof of this. It is a great exercise to think through how to make it completely formal. After doing this on your own, check out Spivak's version of the proof.

"Proof". Suppose $g''(a) > 0$. Then $g'(x)$ is increasing at a. It follows that for all x slightly to the left of a, $g'(x) < g'(a)$, and for all x slightly to the right of a, $g'(x) > g'(a)$. Since $g'(a) = 0$, this means $g'(x) < 0$ for x slightly less than a, and $g'(x) > 0$ for x slightly larger than a. But this implies that g is decreasing to the left of a, and increasing to the right of a. Finally, we deduce that g must have a minimum at a.

A similar argument gives the corresponding result when $g''(a) < 0$. "QED"

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