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UNIVERSITY OF TORONTO SCARBOROUGH

MATA37H3 : Calculus for Mathematical Sciences II

REFERENCE SHEET

The (natural) logarithm and exponential functions

(1) The (natural) logarithm function is defined:

$$\log x = \int_1^x \frac{1}{t}\, dt.$$

(2) For all $a, b > 0$, we have $\log ab = \log a + \log b$.

(3) For all $n \in \mathbb{N}$ and all $x > 0$, we have $\log x^n = n \log x$.

(4) $\exp = \log^{-1}$.

(5) $\exp' = \exp$.

(6) $e = \exp(1)$.

(7) For $a > 0$ and $x \in \mathbb{R}$, we define the function a^x by $a^x = \exp(x \log a)$.

Inverse trigonometric functions

Let

- $f : [-\pi/2, \pi/2] \to [-1, 1]$ by $f(x) = \sin x$,
- $g : [0, \pi] \to [-1, 1]$ by $g(x) = \cos x$, and
- $h : (-\pi/2, \pi/2) \to \mathbb{R}$ by $h(x) = \tan x$.

Then we define the inverse trigonometric functions by

$$\arcsin = f^{-1}, \arccos = g^{-1}, \arctan = h^{-1}.$$

Definition of partitions, upper and lower sums

(1) $\mathcal{P}$ is a *partition* of $[a, b]$ if $\mathcal{P}$ is a finite subset of $[a, b]$ which contains both endpoints.

(2) Given a function f defined on $[a, b]$ and a partition $\mathcal{P}$ of $[a, b]$, set

$$L(f, \mathcal{P}) = \sum_{j \le n} m_j (t_j - t_{j-1})$$

where $\mathcal{P} = \{t_0, t_1, \ldots, t_n\}$ with $t_i < t_{i+1}$ for all i, and $m_j = \inf\{f(x) : x \in [t_{j-1}, t_j]\}$. Similarly, define

$$U(f, \mathcal{P}) = \sum_{j \le n} M_j (t_j - t_{j-1})$$

where $M_j = \sup\{f(x) : x \in [t_{j-1}, t_j]\}$.

(3) For any partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a, b]$, and any function f which is defined and bounded on $[a, b]$, we have

$$L(f, \mathcal{P}) \le U(f, \mathcal{Q}).$$

UNIVERSITY OF TORONTO SCARBOROUGH

MATA37H3 : Calculus for Mathematical Sciences II

FINAL EXAMINATION

April 23rd, 2012

Duration – 3 hours
Aids: none

NAME (PRINT): ______________________________
Last/Surname First/Given Name

STUDENT NO: ____________KEY____________

TUTORIAL: ______________________________
Tutorial section (Number or Schedule) Name of TA

Qn. #	Value	Score
COVER PAGE	5	
1	15	
2	16	
3	8	
4	20	
5	16	
6	20	
Total	100	

TOTAL: ____________

Please read the following statement and sign below:

I understand that any breach of academic integrity is a violation of The Code of Behaviour on Academic Matters. By signing below, I pledge to abide by the Code.

SIGNATURE:______________________________

(1) Prove each of the following assertions. You may freely quote the properties listed on the reference page without proof, but any other properties of $\log$ or $\exp$ must be explicitly proved in your solution. (You are allowed to refer to any theorems from lecture or the text which are *not* about $\log$ or $\exp$.)

(a) (5 points) $\lim_{x\to\infty} \log x = \infty$.

Given M, it suffices to show that $\log x > M$ for all sufficiently large x.

By the Archimedean property, there exists a natural number

$$n > \frac{M}{\log 2}.$$

Since $\log 2 > 0$ (as can be seen from the definition of $\log$), property (3) from the reference sheet implies that

$$\log 2^n = n \log 2 > M.$$

Finally, applying the fundamental theorem of calculus to the definition of the logarithm, we see that $\frac{d}{dx}\log x = \frac{1}{x} > 0$ for all $x > 0$; in particular, $\log x$ is an increasing function. It follows that for all $x > 2^n$,

$$\log x > \log 2^n > M.$$

(b) (5 points) $\lim_{x\to\infty} e^x = \infty$.

First, observe that $\exp$ is a function: $\log$ is increasing, hence is injective, so its inverse is a function. Moreover, since $\log$ is increasing, we conclude that $\exp$ must be increasing as well. It follows immediately that $e > 1$, since $\exp 1 > \exp 0$. Thus we may apply definition (7) to conclude that $e^x = \exp(x)$ for all x, since $\log e = 1$ by property (6).

Given M, it suffices to prove that $e^x > M$ for all sufficiently large x. If $M \leq 0$, then we are immediately done, since $e^x > 1$ for all $x > 0$. If $M > 0$, then for all $x > \log M$ we have

$$e^x = \exp(x) > \exp(\log M) = M$$

since $\exp$ is increasing.

(c) (5 points) $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$.

Since both x^2 and e^x tend to ∞ as $x \to \infty$, we may apply L'Hôpital's rule to deduce:

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}$$

(where we have used Property (5) that $\exp' = \exp$). Since both $2x$ and e^x tend to ∞ with x, we may apply L'Hôpital once more to deduce

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x}.$$

This last quantity clearly tends to 0, since the numerator is constant and the denominator tends to ∞. More precisely, given any $\epsilon > 0$, let $x_0 = \log \frac{2}{\epsilon}$. For all $x > x_0$, we have $e^x > \frac{2}{\epsilon}$, since $\exp$ is increasing. Finally, since $e^x > 0$, we conclude that

$$\left|\frac{2}{e^x} - 0\right| < \epsilon.$$

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(2) For this problem, you may freely refer to any of the properties listed on the reference sheet.

(a) (8 points) Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function, and let $\mathfrak{P}$ denote the set of all partitions of $[a, b]$. Prove that $\sup_{\mathcal{P}\in\mathfrak{P}} L(f, \mathcal{P}) \leq \inf_{\mathcal{P}\in\mathfrak{P}} U(f, \mathcal{P})$.

Fix any partition $\mathcal{Q} \in \mathfrak{P}$. By property (3) on the reference sheet, we see that $U(f, \mathcal{Q})$ is an upper bound on the set $\{L(f, \mathcal{P}) : \mathcal{P} \in \mathfrak{P}\}$. It follows that

$$\sup_{\mathcal{P}\in\mathfrak{P}} L(f, \mathcal{P}) \leq U(f, \mathcal{Q}),$$

since the left side is the *least* upper bound.

In the argument above, the partition $\mathcal{Q} \in \mathfrak{P}$ was arbitrary. Thus, $\sup_{\mathcal{P}\in\mathfrak{P}} L(f, \mathcal{P})$ is a lower bound on the set $\{U(f, \mathcal{Q}) : \mathcal{Q} \in \mathfrak{P}\}$. It follows that

$$\sup_{\mathcal{P}\in\mathfrak{P}} L(f, \mathcal{P}) \leq \inf_{\mathcal{Q}\in\mathfrak{P}} U(f, \mathcal{Q}),$$

since the right hand side is the *greatest* lower bound.

(b) (8 points) Show by example that it's possible for $\sup_{\mathcal{P}\in\mathfrak{P}} L(f, \mathcal{P}) < \inf_{\mathcal{P}\in\mathfrak{P}} U(f, \mathcal{P})$. (You must prove that your example really is an example!)

Let

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

I claim that for all partitions $\mathcal{P} \in \mathfrak{P}$, we have $L(f, \mathcal{P}) = 0$ and $U(f, \mathcal{P}) = 1$. Indeed, suppose $\mathcal{P} \in \mathfrak{P}$, say

$$\mathcal{P} = \{t_0, t_1, \ldots, t_n\}$$

with $t_0 = a$, $t_n = b$, and $t_i < t_j$ whenever $i < j$. Then

$$m_i = \inf_{x\in[t_{i-1},t_i]} f(x) = 0$$

for all i, since $\mathbb{Q}$ is dense. Similarly, since the irrationals are dense, we have

$$M_i = \sup_{x\in[t_{i-1},t_i]} f(x) = 1.$$

It follows that $L(f, \mathcal{P}) = 0$ and $U(f, \mathcal{P}) = 1$ as claimed.

We conclude that

$$\sup_{\mathcal{P}\in\mathfrak{P}} L(f, \mathcal{P}) = 0 \; < \; 1 = \inf_{\mathcal{P}\in\mathfrak{P}} U(f, \mathcal{P}).$$

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(3) (8 points) Determine the exact numerical value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

You must justify your answer to receive credit. [*Hint: Find the Taylor series of* $\arctan x$.]

I first claim that the Taylor series for $\arctan x$ is given by

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (*)$$

One can obtain this in the standard way (write $\arctan x = a_0 + a_1 x + a_2 x^2 + \cdots$, plug in $x = 0$ to find a_0; differentiate both sides and plug in $x = 0$ to find a_1; differentiate both sides and plug in $x = 0$ to find a_2; etc.). Instead, I describe a tricky shortcut.

First, we find the derivative of $\arctan x$. To do this, we evaluate

$$\frac{d}{dx} \tan(\arctan x)$$

in two different ways. On one hand, it simply equals 1 (since $\tan(\arctan x) = x$). On the other hand, by chain and quotient rules,

$$\frac{d}{dx} \tan(\arctan x) = \frac{1}{\cos^2(\arctan x)} \cdot \frac{d}{dx} \arctan x.$$

It follows that

$$\frac{d}{dx} \arctan x = \cos^2(\arctan x) = \frac{1}{1+x^2}$$

Recall the expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

(note that this is the Taylor series of $\frac{1}{1+x}$, and is also the formula for the sum of an infinite geometric series). Plugging in $x = t^2$ yields

$$\frac{d}{dt} \arctan t = \frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots$$

Integrating both sides from 0 to x gives the Taylor expansion claimed in (*).

Plugging in $x = 1$ into the Taylor expansion (*) gives

$$1 - 1/3 + 1/5 - 1/7 + 1/9 - \cdots = \frac{\pi}{4}$$

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(4) (20 points) Suppose that f is integrable on $[a,b]$, and that $f = g'$ for some function g. Prove that
$$\int_a^b f = g(b) - g(a).$$
[*Warning: f might not be continuous on the interval!*]

I first claim that
$$L(f,\mathcal{P}) \le g(b) - g(a) \le U(f,\mathcal{P}) \qquad (\flat)$$
for any partition $\mathcal{P}$ of $[a,b]$. To see this, suppose $\mathcal{P} = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a,b]$, with $t_i < t_j$ whenever $i < j$. By the Mean Value Theorem, for every i there exists $x_i \in [t_{i-1}, t_i]$ such that
$$g'(x_i) = \frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}}.$$
Since $f = g'$, it follows that
$$f(x_i)\big(t_i - t_{i-1}\big) = g(t_i) - g(t_{i-1})$$
for all i. From the definition of $L(f,\mathcal{P})$ and infimum, we see that
$$\begin{aligned} L(f,\mathcal{P}) &= \sum_i \left(\inf_{[t_{i-1},t_i]} f\right) \cdot \big(t_i - t_{i-1}\big) \\ &\le \sum_i f(x_i)\big(t_i - t_{i-1}\big) \\ &= \sum_i \Big(g(t_i) - g(t_{i-1})\Big) \\ &= g(b) - g(a). \end{aligned}$$
Similarly, we see that $U(f,\mathcal{P}) \ge g(b) - g(a)$. Thus, $(\flat)$ is proved.

The definitions of inf and sup immediate imply that
$$\sup_{\mathcal{P}} L(f,\mathcal{P}) \le g(b) - g(a) \le \inf_{\mathcal{P}} U(f,\mathcal{P}).$$
Since f in integrable by hypothesis,
$$\sup_{\mathcal{P}} L(f,\mathcal{P}) = \int_a^b f = \inf_{\mathcal{P}} U(f,\mathcal{P}),$$
so we conclude that
$$\int_a^b f = g(b) - g(a).$$

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(5) Evaluate the integrals below. You must justify your answer to receive credit.

(a) (8 points) $\int_0^1 \arctan x \, dx$.

We first integrate by parts. Set $u = \arctan x$ and $dv = dx$. This gives $du = \frac{dx}{1+x^2}$ and $v = x$, whence

$$\int_0^1 \arctan x \, dx = x \arctan x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx \qquad (\dagger)$$

Our next task is to evaluate the integral on right hand side of (†), which I'll do via u-substitution. Let $u = 1 + x^2$. Then $du = 2x \, dx$, so

$$\int_0^1 \frac{x}{1+x^2} \, dx = \int_1^2 \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2} \log u \Big|_1^2 = \frac{\log 2}{2}$$

Plugging this back into (†) gives

$$\int_0^1 \arctan x \, dx = \frac{\pi}{4} - \frac{\log 2}{2}$$

(b) (8 points) $\int_2^3 \frac{x}{x-1} \, dx$.

We have

$$\begin{aligned} \int_2^3 \frac{x}{x-1} \, dx &= \int_2^3 \frac{x-1+1}{x-1} \, dx \\ &= \int_2^3 \left(1 + \frac{1}{x-1}\right) \, dx \\ &= x \Big|_2^3 + \log(x-1) \Big|_2^3 \\ &= 1 + \log 2. \end{aligned}$$

Alternatively, one can solve this by u-substitution. Letting $u = x - 1$, we have $du = dx$, whence

$$\int_2^3 \frac{x}{x-1} \, dx = \int_1^2 \frac{u+1}{u} \, du = \int_1^2 \left(1 + \frac{1}{u}\right) \, du$$

which gives the same result as before.

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(6) Suppose F is a function satisfying $F(x) = 2$ whenever $0 \le x < 2$, and

$$F(x) = \int_{x/2}^{x^2/2} F \qquad \forall x \ge 2.$$

(In particular, assume that F is integrable on $[x/2, x^2/2]$ $\forall x \ge 2$.)

(a) (7 points) Prove that $F(2) = 2$. [*Careful! There's something to prove here.*]

I claim that F is integrable on $[1, 2]$, and that $\int_1^2 F = 2$. If $F(2) = 2$, we're done. Thus we may suppose (temporarily) that $F(2) \neq 2$. Given $\epsilon > 0$, let

$$\delta = \frac{\epsilon}{2 \cdot \left|F(2) - 2\right|}$$

and consider the partition $\mathcal{P} = \{1, 2-\delta, 2\}$ of the interval $[1, 2]$. Then

$$\begin{aligned} L(F, \mathcal{P}) &= 2\Big((2-\delta) - 1\Big) + \min\{2, F(2)\} \cdot \Big(2 - (2-\delta)\Big) \\ &= 2 + \Big(\min\{2, F(2)\} - 2\Big)\delta \\ &= 2 + \min\Big\{0, \big(F(2) - 2\big)\delta\Big\}. \end{aligned}$$

Similarly, we find

$$U(F, \mathcal{P}) = 2 + \max\Big\{0, \big(F(2) - 2\big)\delta\Big\}.$$

Thus, if $F(2) > 2$ we have $L(F, \mathcal{P}) = 2$ and $U(F, \mathcal{P}) = 2 + \frac{\epsilon}{2}$; otherwise, we have $L(F, \mathcal{P}) = 2 - \frac{\epsilon}{2}$ and $U(F, \mathcal{P}) = 2$. Either way, we conclude that $U(F, \mathcal{P}) - L(F, \mathcal{P}) < \epsilon$. Since $\epsilon > 0$ was arbitrary, this shows that F is integrable on $[1, 2]$.

Now that we know the integral exists, computing it is easy. From above, we see that for all $\epsilon > 0$,

$$2 - \epsilon < L(F, \mathcal{P}) \le \int_1^2 F \le U(F, \mathcal{P}) < 2 + \epsilon.$$

The squeeze theorem then implies that $F(2) = \int_1^2 F = 2$ as claimed.

(b) (7 points) Prove that F is not differentiable at 2.

For F to be differentiable at 2, the left- and right-hand derivatives would have to agree, i.e.

$$\lim_{h \to 0^-} \frac{F(2+h) - F(2)}{h} = \lim_{h \to 0^+} \frac{F(2+h) - F(2)}{h}$$

Since $F(x) = 2$ for all $x \le 2$, the left-hand derivative is 0. I claim that the right-hand derivative is 3, and hence, that F is not differentiable at 2.

To do this, we apply the fundamental theorem of calculus. Let G be an antiderivative of F, say, $G(x) = \int_0^x F$. The fundamental theorem implies that

$$F(x) = \int_{x/2}^{x^2/2} F = G\Big(\frac{x^2}{2}\Big) - G\Big(\frac{x}{2}\Big)$$

for all x such that F is integrable on $[0, x]$. Differentiating and applying chain rule yields

$$\begin{aligned} F'(x) &= xG'\Big(\frac{x^2}{2}\Big) - \frac{1}{2}G'\Big(\frac{x}{2}\Big) \\ &= xF\Big(\frac{x^2}{2}\Big) - \frac{1}{2}F\Big(\frac{x}{2}\Big). \end{aligned}$$

It follows that $F'(2) = 2F(2) - \frac{1}{2}F(1) = 3$ as claimed.

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(c) (6 points) Prove that F is continuous at 2.

We must show that $\lim_{x\to 2} F(x) = F(2)$. Since F is constant to the left of 2, we have $\lim_{x\to 2^-} F(x) = 2$, so it suffices to prove $\lim_{x\to 2^+} F(x) = 2$, i.e.

$$\lim_{h\to 0^+} \int_{(2+h)/2}^{(2+h)^2/2} F = 2.$$

We have

$$\begin{aligned}\int_{(2+h)/2}^{(2+h)^2/2} F &= \int_{1+k}^{2+4k+2k^2} F \qquad\qquad \text{where } k = h/2 \\ &= \int_{1+k}^{2} F + \int_{2}^{2+4k+2k^2} F \\ &= 2(1-k) + \int_{2}^{2+4k+2k^2} F.\end{aligned}$$

Let $m = \inf\{f(x) : x \in [2,3]\}$ and $M = \sup\{f(x) : x \in [2,3]\}$. (Note that these both exist, since f is integrable, and hence bounded, on $[2,3]$.) Then for all $h \in [0,1]$ we have $m \le f(2+h) \le M$, whence

$$m(2h+h^2) = m(4k+4k^2) \le \int_{2}^{2+4k+2k^2} F \le M(4k+4k^2) = M(2h+h^2).$$

We conclude by the squeeze theorem that

$$\lim_{h\to 0^+} \int_{2}^{2+4k+2k^2} F = 0.$$

Also, we have

$$\lim_{h\to 0^+} 2(1-k) = \lim_{h\to 0^+} 2-h = 2.$$

It follows that

$$\begin{aligned}\lim_{x\to 2^+} F(x) &= \lim_{h\to 0^+} F(2+h) \\ &= \lim_{h\to 0^+} \int_{(2+h)/2}^{(2+h)^2/2} F \\ &= \lim_{h\to 0^+} \left(2(1-k) + \int_{2}^{2+4k+2k^2} F\right) \\ &= 2.\end{aligned}$$

Thus the right- and left-hand limits agree, and we conclude that F is continuous at 2.

Total Marks = 100 points