UNIVERSITY OF TORONTO SCARBOROUGH

MATA37H3 : Calculus for Mathematical Sciences II

MIDTERM EXAMINATION # 2

March 9, 2012

Duration – 2 hours
Aids: none

NAME (PRINT): ____________________
Last/Surname First/Given Name

STUDENT NO: KEY

TUTORIAL: ____________________
Tutorial section (Number or Schedule) Name of TA

Qn. #	Value	Score
COVER PAGE	5	
1	20	
2	15	
3	15	
4	15	
5	15	
6	15	
Total	100	

TOTAL: ____________

Please read the following statement and sign below:

I understand that any breach of academic integrity is a violation of The Code of Behaviour on Academic Matters. By signing below, I pledge to abide by the Code.

SIGNATURE:____________________

(1) Suppose f and g are injective functions satisfying the following:

x	1	2	3	4
$f(x)$	2	3	1	4
$f'(x)$	-1	-10	4	
$f''(x)$	-2	3	6	
$g(x)$	3	2	4	1
$g'(x)$	0		-1	
$g''(x)$	-4	8	0	

(a) (5 points) Determine $(g \circ f)^{-1}(2)$. (You must show your work to receive credit.)

First observe that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, so $(g \circ f)^{-1}(2) = f^{-1}\Big(g^{-1}(2)\Big)$. Since g is injective and $g(2) = 2$, we see that $g^{-1}(2) = 2$. Because f is injective and $f(1) = 2$, we have $f^{-1}(2) = 1$. It follows that $f^{-1}\Big(g^{-1}(2)\Big) = f^{-1}(2) = 1$.

(b) (5 points) If $\left.\dfrac{d}{dx}\left(\dfrac{g(x)}{x}\right)\right|_{x=2} = 10$, what is $g'(2)$? (You must show your work to receive credit.)

We have
$$\frac{d}{dx}\left(\frac{g(x)}{x}\right) = \frac{g'(x)x - g(x)}{x^2}.$$
Plugging in $x = 2$ yields
$$\frac{2g'(2) - g(2)}{4} = \frac{2g'(2) - 2}{4} = 10,$$
whence $g'(2) = 21$.

(c) (10 points) Prove that $f \circ g$ has a local maximum at 1. (You must show your work to receive credit.)

We will accomplish this by showing that $(f \circ g)'(1) = 0$ and $(f \circ g)''(1) < 0$. By chain rule, we have
$$(f \circ g)'(a) = f'\Big(g(a)\Big)g'(a). \qquad (*)$$
Plugging in $a = 1$ gives $(f \circ g)'(1) = f'(3)g'(1) = 0$ as claimed.

It now remains only to show that the second derivative is negative at 1. Differentiating (*) by chain rule and product rule, we find
$$(f \circ g)''(a) = f''\Big(g(a)\Big)g'(a)^2 + f'\Big(g(a)\Big)g''(a).$$
It follows that $(f \circ g)''(1) = f''(3)g'(1)^2 + f'(3)g''(1) = -16$. Since this is negative, we conclude that $f \circ g$ has a local maximum at 1.

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(2) (15 points) Suppose f and g are functions defined on all of $\mathbb{R}$, and suppose there is a number $a \in \mathbb{R}$ such that $f(a) = 0$, f is differentiable at a, *and g is continuous at a*. Prove that $(f \cdot g)'(a) = (f' \cdot g)(a)$. [*Warning: g might not be differentiable at a, so the product rule does not apply!*]

By definition of derivative, we have

$$\begin{aligned}
(f \cdot g)'(a) &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a+h)}{h} && \textit{since } f(a) = 0 \\
&= \left(\lim_{h\to 0} \frac{f(a+h)}{h}\right) \cdot \left(\lim_{h\to 0} g(a+h)\right) \\
&= \left(\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}\right) \cdot g(a) && \textit{since } g \textit{ is continuous at } a \\
&= f'(a)g(a) \\
&= (f' \cdot g)(a).
\end{aligned}$$

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(3) (15 points) Suppose f is a function which is differentiable at a, and set $g(x) = f(x)^2$. Without using the chain rule *or product rule*, prove that $g'(a) = 2f(a)f'(a)$.

By definition of the derivative, we have

$$\begin{aligned}
g'(a) &= \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)^2 - f(a)^2}{h} \\
&= \lim_{h\to 0} \frac{\Big(f(a+h) - f(a)\Big)\Big(f(a+h) + f(a)\Big)}{h} \\
&= \left(\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}\right) \cdot \left(\lim_{h\to 0} f(a+h) + f(a)\right) \\
&= f'(a) \cdot \left(\lim_{h\to 0} f(a+h) + \lim_{h\to 0} f(a)\right) \\
&= f'(a) \cdot 2f(a) \qquad \textit{since } f \textit{ is differentiable at } a\textit{, and hence is continuous there} \\
&= 2f(a)f'(a).
\end{aligned}$$

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(4) (15 points) Suppose f is defined and differentiable for all $x > 0$, and $\lim_{x\to\infty} f'(x) = 0$. Set $g(x) = f(x+1) - f(x)$. Prove that $\lim_{x\to\infty} g(x) = 0$.

By the Mean Value Theorem, for all $x > 0$ there exists $c_x \in (x, x+1)$ such that

$$f'(c_x) = \frac{f(x+1) - f(x)}{(x+1) - x} = g(x). \qquad (\dagger)$$

Given $\epsilon > 0$, we know there exists x_0 such that $|f'(x)| < \epsilon$ for every $x > x_0$. It follows that $|f'(c_x)| < \epsilon$ for all $x > x_0$, since $c_x > x$ for all x (and is therefore also larger than x_0). Combining this with equation (†) shows that $|g(x)| < \epsilon$ for all $x > x_0$. Thus, we have proved that $\lim_{x\to\infty} g(x) = 0$ as claimed.

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(5) (15 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

For any $a \in \mathbb{R}$ and any $h \neq 0$, our hypothesis implies

$$0 \leq \left| \frac{f(a+h) - f(a)}{h} \right| \leq |h|.$$

It follows from the squeeze theorem that

$$\lim_{h \to 0} \left| \frac{f(a+h) - f(a)}{h} \right| = 0,$$

which in turn implies that

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = 0.$$

We thus conclude that for any $a \in \mathbb{R}$, f is differentiable at a and $f'(a) = 0$. Therefore, f is constant.

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(6) Let $f(x) = x^2 + \sin x$.

(a) (7 points) Prove that f has no local maxima or minima in the interval $\left(\frac{1}{2}, \infty\right)$.

For any $x > 1/2$, we have

$$-2x < -1 \leq \cos x.$$

In particular, $f'(x) = 2x + \cos x > 0$ for all $x > 1/2$. It follows that $f'(a) \neq 0$ at any $a > 1/2$, so f cannot have a local maximum or minimum at any $a > 1/2$.

(b) (8 points) How many local maxima does f have on all of $\mathbb{R}$? How many local minima does it have? Justify your answers. [*Hint: consider* f''.]

From part (a), we know $f'(x) = 2x + \cos x$. Since f' is continuous everywhere and $f'(-1) < 0 < f'(1)$, the Intermediate Value Theorem implies that f' has at least one root in the interval $(-1, 1)$.

On the other hand, consider $f''(x) = 2 - \sin x$. Since $\sin x \leq 1$ for all x, we see that $f''(x) > 0$ for all x. This implies that $f'(x)$ is always increasing, and can therefore have at most one root. Combining this with the conclusion of the previous paragraph, we see that f' has precisely one root, and that this root occurs somewhere in the interval $(-1, 1)$. Call this unique root, a (i.e. a is the only input such that $f'(a) = 0$).

Since $f''(a) > 0$, we see that f has a local minimum at a. There can be no other local minima or maxima, since $f'(x) \neq 0$ whenever $x \neq a$. We conclude that f has precisely one local minimum, and no local maxima. (Incidentally, this also shows that f has a minimum at a, not just a local minimum.)

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Total Marks = 100 points