# ADDITIVE COMBINATORICS: FINAL PROJECT

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In lieu of a final exam, you will write an expository paper on an interesting theorem or question related to additive combinatorics. The paper should be aimed at your peers, and should be easy to read for your classmates. Your goal is give as much insight into the theorem and the proof as you can in 5–10 pages. The paper should explain what the theorem says, give some motivation, and outline the proof. Some examples of expository mathematical writing can be found on my webpage:

`www.math.toronto.edu/lgoldmak/`

Here are some common ways to write about a theorem. For motivation, you might talk about the history of problem, you might give an application of the theorem, or you might compare it to other theorems. If the statement of the theorem is hard to absorb, you could explain what it says in a particular example. (It is always a great exercise to generate your own examples; in the end, you may decide that the 'well-known' examples are better than the ones you came up with, but it's also possible that your examples will be more interesting, and in any event your work will allow you to better appreciate the theorem and existing examples.) You could also give counterexamples which show why all the hypotheses of the theorem are necessary or why the conclusion is not stronger. If there are open problems related to the theorem, you could mention those. Then you should write something about the proof – I don't necessarily need all the details, but I do require that you at least write a clear outline of the argument. Some common ways to try to understand a proof are to prove a special case of the theorem in which the ideas are all present but some technical aspects are suppressed (e.g. our proof of Ruzsa's theorem for groups of finite exponent). Alternatively, you could prove something in a similar spirit but more elementary. Conversely, you could try to explain why the proof is difficult - why isn't there an easier proof? It's up to you to figure out which of these options gives the most insight into your theorem / problem. You may pick your own topic; I've included a list of possible topics on the next page, in case you need some inspiration. In terms of researching your topic, two websites which are particularly helpful are `www.arxiv.org` and `www.ams.org/mathscinet/` (the latter is only usable from a U of T network).

The paper must be *typed up* and submitted by email to me at `lgoldmak@math.toronto.edu` The deadline for submission is April 19th. A particularly convenient way to type mathematical text is to use LaTeX. If you wish to try it, I would be happy to help you set it up. To get started, you need to download miktex (it's a free download, although it is a very large file and takes some time to download). Then I can send you some sample LaTeXfiles for you to play around with. There are a lot of excellent resources available online to help you, as well.

On the next pages are a list of possible topics, but you should feel free to find your own; any topic which inspires you is fine, so long as it's at least tangentially related to additive combinatorics.

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I am indebted to Larry Guth, who gave a similar assignment in his graduate analysis course at U of T.

SOME POSSIBLE TOPICS:

(1) Higher order Fourier analysis (see Balazs Szegedy's paper on arxiv)
(2) van der Waerden's theorem, the Hales-Jewett theorem, and monochromatic subsets. An open problem in this area is: color the integers using a finite number of colors. Must there exist integers $x$ and $y$ such that $x + y$ and $xy$ are the same color? The answer is yes: $x = y = 2$. But if you exclude this trivial example, it's an open question. You may find Soundararajan's course notes on additive combinatorics (freely available on his Stanford homepage) helpful as a starting point.
(3) Szemerédi-Trotter and other incidence theorems from geometry, and their applications to sum-product. (Zeev Dvir has a long write-up on the subject, which you could use as a starting point.)
(4) Applications of sum-product theorems to computer science. (You may wish to look at the expositions on the subject by Shachar Lovett, Emanuele Viola, and the Princeton mini-course on Additive Combinatorics.)
(5) More sums than differences – when does a set $A$ satisfy $|A + A| > |A - A|$? See recent papers (on the arxiv) by Steven J. Miller et al.
(6) The multiplication table problem and generalizations; see the PhD thesis of Dimitris Koukoulopoulos (available on his homepage), as well as work of Kevin Ford.
(7) Sárkőzy's conjecture on non-decomposability of quadratic residues as a sumset; see the recent paper on the arxiv by Blackburn, Konyagin, and Shparlinski for a starting point. (But there has been a lot of other work on the subject you should look at.)
(8) Sidon sets (e.g. see the paper *Combinatorial problems in finite fields and Sidon sets* by Javier Cilleruelo).
(9) The generalization (by Ben Green and Imre Ruzsa) of Freiman's theorem to arbitrary abelian groups
(10) Theorem: Every large subset of $\mathbb{Z}/p\mathbb{Z}$ can be written in the form $A + A$. This is originally a result of Ben Green; see *Large sets in finite fields are sumsets* by Noga Alon for more current results.
(11) Khovanskii's theorem: given any finite $A \subseteq \mathbb{Z}$, there exists a polynomial $f_A(x)$ such that $|nA| = f_A(n)$ for all sufficiently large $n$. See the paper by Jelinek and Klazar on the arxiv, or Ruzsa's notes *Sumsets and Structure*.
(12) The Green-Tao theorem: there exist arbitrarily long arithmetic progressions of primes. See the recent exposition by David Conlon, Jacob Fox, and Yufei Zhao on the arxiv.
(13) The Breuillard-Green-Tao theorem: a generalization of Freiman's theorem to arbitrary (non-abelian) groups. (See their paper.)
(14) Harald Helfgott's work on $SL_2$; see section 4 of Ben Green's survey paper *Approximate groups and their applications* on the arxiv.
(15) Bourgain's proof of the existence of long arithmetic progressions in $A + B$. (You might find Olof Sisask's expository paper of the same name to be a good starting point.)
(16) Sárkőzy's theorem on perfect squares. (See Neil Lyall's paper *A new proof of Sarkozy's theorem* for references on the problem.)
(17) Roth's theorem on three term progressions and Gowers' generalization to four term progressions. (You may wish to see Soundararajan's course notes on additive combinatorics, available on his homepage at Stanford.)

(18) The ergodic proof of Szemerédi's theorem; see the paper in *Bulletin of the AMS* by H. Furstenberg, Y. Katznelson, and D. Ornstein. (This is one of the most beautiful papers known to me in any field of mathematics.)
(19) Applications of the Balog-Szemerédi-Gowers theorem.
(20) Sum-product theorems: Solymosi's work over $\mathbb{R}$ (we presented his strongest result in class, but his earlier work is takes a different approach which is also interesting); the extension to $\mathbb{C}$ by S. V. Konyagin and M. Rudnev (see their paper on the arxiv); and work on sum-product over finite fields, for example *Estimates for the number of sums and products and for exponential sums in fields of prime order* by J. Bourgain, A. Glibichuk, and S. V. Konyagin, and *A sum-product estimate in finite fields, and applications* by J. Bourgain, N. Katz, and T. Tao.
(21) The Kakeya conjecture over finite fields was seen to be extremely difficult, until Zeev Dvir (a graduate student in computer science) discovered a beautifully simple argument; see his paper *On the size of Kakeya sets in finite fields*, available on the arxiv. Also see the follow-up paper by S. Saraf and M. Sudan, *Improved lower bound on the size of Kakeya sets over finite fields*.
(22) Dvir's work helped to popularize the 'polynomial method,' a powerful technique which is useful both throughout math and computer science. Discuss the polynomial method. You may find the MIT course given by Larry Guth (course notes freely available online) helpful. One of the spectacular applications of the method is the near-resolution of the notorious Erdős distinct distances problem; see the paper by L. Guth and N. Katz.
(23) Sanders' work on the polynomial Freiman-Ruzsa conjecture. In addition to Sanders' original paper, you may find S. Lovett's expository paper on the subject helpful.
(24) Applications of the polynomial Freiman-Ruzsa conjecture to computer science. You may wish to start by exploring *An additive combinatorics approach relating rank to communication complexity* by E. Ben-Sasson, S. Lovett, and N. Ron-Zewi, and *From affine to two-source extractors via approximate duality* by N. Ron-Zewi and E. Ben-Sasson.

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