LECTURE 13: SUMMARY

There was some concern about the proof from last lecture that any two cardinals are comparable; since $\mathbb{R}$ and $\mathbb{N}$ are both well-ordered, can't we use the same technique as in that theorem to prove that $\mathbb{R} \hookrightarrow \mathbb{N}$? We discussed why this isn't the case. This completes our discussion of set theory for the semester.

We then took up our next subject: sequences and series. We start by discussing sequences. Given a function $f : \mathbb{N} \to \mathbb{R}$, let $a_n = f(n)$ for each $n \in \mathbb{N}$. We call this ordered collection of real numbers a *sequence*, and denote it by (a_n). Note that the sequence (a_n) is different from the set of values $\{a_n\}$, since a sequence is ordered while a set is not.

Remark. *By our definition, every sequence (a_n) must be infinite. On the other hand, the range of values $\{a_n\}$ might be finite. For example, the sequence $(2, 2, 2, \dots)$ is an infinite sequence, whose range consists of the single number $\{2\}$.*

We next discussed convergence and divergence of a sequence. (This is rather similar to limits of functions, and so should be fairly familiar.) We say a sequence (a_n) *converges to A* (denoted: $\lim_{n\to\infty} a_n = A$) iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - A| < \epsilon$ whenever $n > N$. Sometimes we will be lazy, and simply say that (a_n) converges (without specifying what it converges to). If a sequence does not converge to any real number, it is said to *diverge*. We gave several examples of convergent and divergent sequences.

We next turned to infinite series. First, recall 'sigma' notation: given a sequence (a_n), define

$$\sum_{n=k}^{\ell} a_n = a_k + a_{k+1} + \cdots + a_\ell.$$

(This sum is also denoted $\sum_{k\le n\le \ell} a_n$.) We now define the symbol $\sum_{n=1}^{\infty} a_n$, called an *infinite series*. Despite the notation, it is important to note that this is NOT the sum of infinitely many numbers, since it's not clear what that even means! Instead, we define it as follows. For every $N \in \mathbb{N}$, set

$$S_N := \sum_{n=1}^{N} a_n,$$

and consider the sequence (S_N). This is often called the sequence of *partial sums*. If this sequence converges to a real number S, we say that the infinite series converges to S, and write

$$\sum_{n=1}^{\infty} a_n = S.$$

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Once again, let me emphasize that this does NOT mean that the sum of all the a_n's equals S; it means that the limit of partial sums converges to S. If the sequence of partial sums doesn't converge to any real number, we say the infinite series diverges. Note that an infinite series may diverge because the partial sums get really huge (e.g. $\sum_{n=1}^{\infty} 1$), but there are other ways the series might diverge (e.g. $\sum_{n=1}^{\infty} (-1)^{n-1}$).

We then discussed some examples of converging and diverging series. Perhaps the most important (and familiar) example we discussed was the *geometric series*:

$$\sum_{n=1}^{\infty} x^n. \tag{*}$$

By examining the partial sums

$$\sum_{n=1}^{N} x^n = \frac{x}{1-x}(1-x^N)$$

we were able to prove that the geometric series (*) converges if $|x| < 1$, and that it diverges if $|x| > 1$. Finally, by considering the two cases $x = \pm 1$, we concluded that (*) converges iff $|x| < 1$.