LECTURE 15: SUMMARY

Today we continued exploring infinite series, in particular proving two important convergence tests: the root test and the ratio test.

Theorem 1 (Root Test)**.** *Suppose* (a_n) *is a sequence of non-negative real numbers, and that* $\alpha := \lim_{n\to\infty} a_n^{1/n}$ *exists.*

- *If* $\alpha < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *converges.*
- *If* $\alpha > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Before proving this, we talked about what happens when $\alpha = 1$. David pointed out that $\sum \frac{1}{n}$ diverges while $\sum \frac{1}{n^2}$ converges, but for both series $\alpha = 1$. This shows that the Root Test is too crude to give information in the case $\alpha = 1$.

Proof. First, the case $\alpha < 1$. Pick $\beta \in (\alpha, 1)$. From the definition of α, it follows that there exists a large K such that $a_n^{1/n} < \beta$ for all $n > K$. It follows that for all $M > K$,

$$0 \leq \sum_{K<n\leq M} a_n < \sum_{K<n\leq M} \beta^n.$$

Let

$$S_N := \sum_{n=1}^{N} a_n.$$

I claim that the sequence (S_N) is bounded and monotonic (and hence converges, by the MCT). Monotonicity is clear, since $a_n \geq 0$ for all n by hypothesis. So, it suffices to prove boundedness. First note that S_K is some constant (since K is constant), and that we have $0 \leq S_N \leq S_K$ for all $N \leq K$. For $N > K$, we have

$$0 \leq S_N = S_K + \sum_{K<n\leq N} a_n < \sum_{K<n\leq N} \beta^n.$$

From a prior lecture, we know that the geometric series

$$\sum_{n=1}^{\infty} \beta^n$$

converges. It follows that partial sums of this geometric series are bounded. But this implies that

$$\sum_{K<n\leq N} \beta^n$$

Date: March 5th, 2013.

are bounded for all $N \geq K$, as well. (Make sure you understand why!) We have therefore shown that (S_N) is bounded and monotone; we conclude that it converges.

We now consider the case $\alpha > 1$. Then $a_n^{1/n} > 1$ for all sufficiently large n, whence $a_n > 1$ for all large n. It follows that $a_n \not\to 0$ as $n \to \infty$, so the series cannot converge. $\square$

The above theorem is nice in principle, but somewhat awkward to apply in practice (taking nth roots can be tricky). We now state a convergence test which is weaker, but more user-friendly.

Theorem 2 (Ratio Test)**.** *Suppose (a_n) is a sequence of non-negative real numbers, and that $\alpha := \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists.*

- *If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.*
- *If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof. We first consider the case $\alpha < 1$. As in the proof of the root test, pick $\beta \in (\alpha, 1)$. From the definition of α, there exists N such that

$$\frac{a_{n+1}}{a_n} < \beta$$

for all $n \geq N$. It follows that

$$\begin{aligned} a_{N+1} &< \beta a_N \\ a_{N+2} &< \beta a_{N+1} < \beta^2 a_N \\ a_{N+3} &< \beta a_{N+2} < \beta^3 a_N \end{aligned}$$

and, more generally, that for all $M > N$,

$$a_{N+k} < \beta^k a_N.$$

Thus, for all $K \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{n=1}^{N+K} a_n &= \sum_{n\leq N} a_n + \sum_{N<n\leq N+K} a_n \\ &< \sum_{n\leq N} a_n + a_N \sum_{k\leq K} \beta^k \end{aligned}$$

The first sum is a constant, while the second is a geometric series. We therefore conclude that the sequence of partial sums

$$S_M := \sum_{n=1}^{M} a_n$$

is bounded. It is also monotonic (since $a_n \geq 0$). The MCT now implies that (S_M) converges.

Next, consider the case $\alpha > 1$. In this case, there exists N such that $a_{n+1} > a_n$ for all $n \geq N$. In particular, $a_n \not\to 0$ as $n \to \infty$, so the series cannot converge. $\square$