LECTURES 16 AND 17: SUMMARY

In these two lectures, we proved a number of fundamental results about convergence of sequences and series. We started with an easy observation: that if a sequence converges to some number, then the terms of the sequence are eventually all close to one another. More precisely, we proved the following.

Proposition 1. *If a sequence (a_n) converges, then for any $\epsilon > 0$ there exists N such that*

$$|a_n - a_m| < \epsilon$$

for all $m, n \geq N$.

Proof. Given $\epsilon > 0$. Let

$$A := \lim_{n\to\infty} a_n.$$

There exists N such that for all $n \geq N$,

$$|a_n - A| < \epsilon/2.$$

Thus, for all $m, n \geq N$,

$$|a_n - a_m| = |(a_n - A) + (A - a_m)| \leq |a_m - A| + |a_n - A| < \epsilon. \qquad \square$$

Remarkably, the converse of this proposition is also true. To state it in a cleaner way, we made a definition:

Definition. *A sequence (a_n) is said to be a* Cauchy sequence *iff for any $\epsilon > 0$ there exists N such that*

$$|a_n - a_m| < \epsilon$$

for all $m, n \geq N$.

In other words, a Cauchy sequence is one in which the terms eventually cluster together. We will prove (over the course of $2 + \epsilon$ lectures) the following theorem:

Theorem 2 (Cauchy Criterion)**.** *A sequence is Cauchy iff it converges.*

Why is this useful? Usually when we explore the convergence of a sequence, we first guess whether or not it converges (and what it converges to), and then verify the guess with an $\epsilon - \delta$ proof. This approach works for nice sequences, but often it's not clear how to guess about the convergence. The Cauchy Criterion allows us to shift from an external point of view – one in which we know not only the sequence, but also the limit of that sequence – to an internal one, where we can decide convergence based purely on the behavior of the sequence itself.

One nice example of this is the construction of $\mathbb{R}$. One way of doing this is to consider all Cauchy sequences consisting of rational numbers. Every such Cauchy sequence converges to *something*,

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but this something might be irrational. To every Cauchy sequence of rational numbers, we associate a symbol α (intuitively, α is the limit of the sequence). We can define addition and multiplication of real numbers in terms of operations on the underlying Cauchy sequences, and thus construct all of $\mathbb{R}$. Although this is actually a bit painful to carry out rigorously, the underlying idea is elegant and straightforward, and what makes it work is that we know that Cauchy sequences converge.

Before proving Theorem 2, we prove the following nice consequence:

Corollary 3. *If the series* $\sum_{n=1}^{\infty} |a_n|$ *converges, then so does* $\sum_{n=1}^{\infty} a_n$.

If $\sum_{n=1}^{\infty} |a_n|$ converges, we say that the series $\sum_{n=1}^{\infty} a_n$ *converges absolutely*. Thus, the above corollary asserts that if a series converges absolutely, then it converges. It turns out that the converse to this is false, as we shall see soon.

Proof of Corollary. Let

$$A_N := \sum_{n \le N} |a_n| \qquad \text{and} \qquad S_N := \sum_{n \le N} a_n.$$

We are given that (A_N) converges, and wish to prove that (S_N) does as well.

Given $\epsilon > 0$. Since (A_N) converges, Theorem 2 implies that (A_N) is Cauchy, whence $\exists K$ such that

$$|A_N - A_M| < \epsilon$$

for all $M, N > K$. Pick any $M, N > K$; without loss of generality, we have $N \ge M$. Then

$$|S_N - S_M| = |a_{M+1} + a_{M+2} + \cdots + a_N| \le |a_{M+1}| + |a_{M+2}| + \cdots + |a_N| = |A_N - A_M| < \epsilon.$$

Thus, (S_N) is Cauchy. The Cauchy Criterion implies that it converges. □

Thus, the Cauchy Criterion makes this result rather trivial to prove. I challenge the reader to come up with an alternative proof of the Corollary, without relying on the Cauchy Criterion. This can be accomplished (you'll see one approach on the assignment), but it isn't easy.

We are now ready to prove the Cauchy Criterion.

Proof of Theorem 2. We've already proved that if a sequence converges, it is Cauchy. It therefore suffices to prove that a Cauchy sequence (a_n) must converge. The tricky part is that we have no idea, *a priori*, what it converges to! This is what makes the proof challenging.

The proof proceeds in several steps, which we isolate and prove subsequently.

Step 1. Since (a_n) is Cauchy, it must be bounded.
Step 2. Since (a_n) is bounded, it has a convergent subsequence (a_{n_k}). Let $x := \lim_{k\to\infty} a_{n_k}$.
Step 3. $\lim_{n\to\infty} a_n = x$. □

This is the big picture of the proof. We now justify the individual steps. Steps 1 and 3 are fairly straightforward; Step 2 will take some more effort. We proceed in order.

Proposition (Step 1)**.** *If a sequence (a_n) is Cauchy, then it is bounded.*

Proof. There exists N such that $|a_n - a_m| < 1$ for all $n, m \geq N$. In particular, we have $|a_n - a_N| < 1$ for all $n \geq N$, whence

$$a_n \in (a_N - 1, a_N + 1)$$

for all $n \geq N$. Thus, $\{a_n : n \geq N\}$ is bounded. Also, $\{a_n : n < N\}$ is bounded (since it's finite). We conclude that the entire range of the sequence $\{a_n\}$ is bounded, as claimed. □

Our proof of Step 2 will rely on the following result:

Theorem (Monotone Subsequence Theorem)**.** *Every sequence has a monotone subsequence.*

We will prove this in the next lecture. For the time being, we take it on faith and prove Step 2.

Theorem (Step 2, aka Bolzano-Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

Proof. Suppose (a_n) is a bounded sequence. By the Monotone Subsequence Theorem, it has a monotone subsequence (a_{n_k}). But then this subsequence is both bounded and monotone, whence (by the MCT) it is convergent. □

Finally, we prove Step 3:

Proposition (Step 3)**.** *If a subsequence of a Cauchy sequence converges to x, then the sequence itself converges to x.*

Proof. Let (a_n) be a Cauchy sequence, and let (a_{n_k}) be a convergent subsequence. Set

$$x = \lim_{k \to \infty} a_{n_k}.$$

We wish to prove that $a_n \to x$ as $n \to \infty$.

Given $\epsilon > 0$. There exists K such that for all $k > K$,

$$|a_{n_k} - x| < \epsilon/2.$$

Also, since (a_n) is Cauchy, there exists N such that

$$|a_n - a_m| < \epsilon/2$$

for all $m, n > N$. Pick $\ell > K$ large enough so that $n_\ell > N$. Then for all $n > N$, we have

$$|a_n - x| = |(a_n - a_{n_\ell}) + (a_{n_\ell} - x)| \leq |a_n - a_{n_\ell}| + |a_{n_\ell} - x| < \epsilon$$

which concludes the proof. □

Thus to conclude the proof of the Cauchy Criterion, it remains only to prove the Monotone Subsequence Theorem. We will do this at the start of next lecture.