LECTURE 19: SUMMARY

Recall from last time the following definition.

Definition (Metric). *Given a nonempty set X, we say that $d : X \times X \to \mathbb{R}$ is a* distance *(or* metric*) on X iff it satisfies the following three properties:*

(1) $d(x, y) = 0$ *iff* $x = y$*;*
(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*; and*
(3) $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in X$*. (Triangle Inequality)*

A *metric space* is a set X, together with a metric d on X. As we shall see later, every set is metrizable, i.e. given any set, there exists a metric on it. Moreover, we can often find more than one metric on a given space. Thus, when talking about a metric space, it's important to give both the space *and* the particular metric we have in mind.

Note that nowhere in the definition is the distance between two points required to be non-negative. This is a bit unsettling, since we are not used to dealing with negative distances. Fortunately, we don't have to:

Proposition 1. *Given a metric space (X, d). Then $d(x, y) \geq 0$ for all $x, y \in X$.*

Proof. We have $d(x, y) = \frac{1}{2}\big(d(x, y) + d(y, x)\big) \geq \frac{1}{2}d(x, x) = 0$. □

We then started going through some examples of metric spaces.

(i) The real line $\mathbb{R}$, with metric $d(x, y) := |x - y|$. This is a very familiar example.

(ii) n-dimensional space $\mathbb{R}^n$, with the "standard" (or "Euclidean") metric

$$d(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}.$$

(We developed this by first writing figuring out formulas in the cases $n = 2$ and $n = 3$.) It's immediately clear that this satisfies properties (1) and (2) of a metric, but it is not at all clear that it satisfies the triangle inequality. In today's lecture, we demonstrated that if

$$(*) \qquad \left| \sum_i a_i b_i \right| \leq \left(\sum_i a_i^2 \right)^{1/2} \left(\sum_i b_i^2 \right)^{1/2}$$

for all a_i, b_i, then the Euclidean metric satisfies the triangle inequality. The inequality (*) has many other applications as well, and for this reason carries a name: it is called the *Cauchy-Schwarz inequality*. We will prove it next lecture.

Date: March 19th, 2013.

(iii) n-dimensional space $\mathbb{R}^n$, with the "taxicab" metric

$$d(x, y) := \sum_{i=1}^{n} |x_i - y_i|$$

This was much easier to verify as a metric, but what exactly is it measuring? The reason it's called the taxicab metric is the following: imagine you're in a city. The distance between two points isn't measured in a straight line in the city; rather it's measured in how blocks it takes you to get from the first point to the second. Hence, the name.

The third example shows that the same space might have multiple metrics on it. We will see further examples of this next lecture.