LECTURE 20: SUMMARY

We started by proving the Cauchy-Schwarz inequality:

$$\left|\sum_i a_i b_i\right| \le \left(\sum_i a_i^2\right)^{1/2} \left(\sum_i b_i^2\right)^{1/2}$$

for all a_i, b_i. Actually, we proved a rather stronger result:

Theorem 1 (Hölder's inequality)**.** *Given* $p, q > 0$ *such that* $1/p + 1/q = 1$. *Then*

$$(\dagger) \qquad \left|\sum_i a_i b_i\right| \le \left(\sum_i a_i^p\right)^{1/p} \left(\sum_i b_i^q\right)^{1/q}.$$

Proof. Throughout, ETS shall stand for Enough To Show.

By the triangle inequality, ETS

$$\sum_n |a_n b_n| \le \left(\sum_k |a_k|^p\right)^{\frac{1}{p}} \left(\sum_k |b_k|^q\right)^{\frac{1}{q}}$$

Let $A_n = |a_n|^p$ and $B_n = |b_n|^q$. So ETS

$$\sum_n A_n^{1/p} B_n^{1/q} \le \left(\sum_k A_k\right)^{\frac{1}{p}} \left(\sum_k B_k\right)^{\frac{1}{q}}$$

whence ETS

$$\sum_n \left(\frac{A_n}{\sum A_k}\right)^{\frac{1}{p}} \left(\frac{B_n}{\sum B_k}\right)^{\frac{1}{q}} \le 1$$

Let $x = 1/p$. Then $1/q = 1 - x$, so ETS that for all $x \in [0, 1]$,

$$\sum_n \left(\frac{A_n}{\sum A_k}\right)^{x} \left(\frac{B_n}{\sum B_k}\right)^{1-x} \le 1$$

Let

$$\alpha_n = \frac{A_n}{\sum A_k} \qquad \text{and} \qquad \beta_n = \frac{B_n}{\sum B_k}$$

Then ETS that $\forall x \in [0, 1]$ and $\forall \alpha_n, \beta_n$ positive such that $\sum \alpha_n = \sum \beta_n = 1$,

$$f(x) := \sum_n \beta_n \left(\frac{\alpha_n}{\beta_n}\right)^x \le 1$$

Date: March 21st, 2013.

Since $f(0) = f(1) = 1$, ETS that $f(x)$ is concave up on $[0, 1]$. Since $\beta_n > 0$, ETS

$$f_n(x) := \left(\frac{\alpha_n}{\beta_n}\right)^x$$

is concave up on $[0, 1]$, whence ETS that $g(x) := a^x$ is concave up on $[0, 1]$, where a is any positive constant. Finally, we're done ETSing: $g''(x) = a^x (\log a)^2 \geq 0$. $\square$

Taking $p = q = 2$ immediately yields the Cauchy-Schwarz inequality. This, in turn, implies that the Euclidean metric satisfies the triangle inequality (and hence, deserves the title 'metric').

Recall from last time the following examples of metric spaces.

(i) The real line $\mathbb{R}$, with metric $d(x, y) := |x - y|$.

(ii) n-dimensional space $\mathbb{R}^n$, with the "standard" (or "Euclidean") metric

$$d(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}.$$

(iii) n-dimensional space $\mathbb{R}^n$, with the "taxicab" metric

$$d(x, y) := \sum_{i=1}^{n} |x_i - y_i|.$$

Today we added a few more to the list:

(iv) n-dimensional space $\mathbb{R}^n$, with the "chessboard" metric

$$d(x, y) := \max_i |x_i - y_i|.$$

This was relatively straightforward to verify as a metric, although checking triangle inequality required some notation. I then also hinted at where the name comes from: consider a chessboard, and define the distance between two squares to be the fewest number of steps required for a king to travel from one to the other.

(v) n-dimensional space $\mathbb{R}^n$, with the "British Rail" metric

$$d(x, y) := |x| + |y|$$

This actually isn't a metric as written (why not?); fortunately, we will be able to patch it up next lecture and make it a metric. The name comes from the fact that to travel from any city to any other in the UK, one must go through London. Similarly, this metric measures the distance between x and y by measuring the distance from x to the 'origin', and then the distance from the origin to y.

(vi) Any non-empty set X, with the "discrete" metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

This metric is not particularly refined, but demonstrates that any space is metrizable.

(vii) The space $\mathcal{C}_{[0,1]} := \{f : [0,1] \to \mathbb{R},\ \text{a continuous function}\}$, with the metric

$$d(f,g) = \max_{t \in [0,1]} |f(t) - g(t)|.$$

We will discuss this example more next lecture.